CINEFORUM
ISSN: 0009-7039
Vol. 66. No. S5, 2026 (Special Issue 5)

Strong Interval-Valued Intuitionistic Fuzzy Ideals of Semirings.

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Abstract
This research presents the idea of strong interval-valued intuitionistic fuzzy ideals within semirings and explores their key algebraic characteristics. This concept builds upon the existing theory of interval-valued intuitionistic fuzzy ideals by imposing more stringent conditions on the interval-valued membership and non-membership functions when considering addition and multiplication. To validate this concept, several fundamental properties, examples, and characterizations have been established. Additionally, significant structural findings are demonstrated, such as closure under intersections, behaviour with respect to semiring homomorphisms and isomorphisms, level subset characterizations, and properties related to finite sums and products. These findings lay a robust theoretical groundwork for the exploration of strong interval-valued intuitionistic fuzzy ideals and enhance the field of interval-valued intuitionistic fuzzy algebra, with potential applications in fuzzy algebraic systems, decision-making, and modelling uncertainty....
Keywords: Semiring, Interval-valued intuitionistic fuzzy set, Strong interval-valued intuitionistic fuzzy ideal, Fuzzy ideal...

INTRODUCTION

Fuzzy set theory, as introduced by Zadeh [13], offers a robust mathematical framework for addressing uncertainty and vagueness across various scientific and engineering fields. To address the constraints of traditional fuzzy sets, Atanassov developed intuitionistic fuzzy sets by integrating both membership and non-membership functions, thus offering a more adaptable representation of uncertain data than classical fuzzy sets. Subsequently, Atanassov and Gargov expanded this idea to interval-valued intuitionistic fuzzy sets (IVIFSs), where membership and non-membership values are expressed as intervals rather than exact numbers, enhancing the model's suitability for scenarios with incomplete or imprecise data [2]. Further theoretical advancements in interval-valued intuitionistic fuzzy operators have bolstered the theory's relevance in algebra and decision science [1]. In recent years, interval-valued intuitionistic fuzzy sets have found applications in optimization, decision-making, artificial intelligence, information systems, and pattern recognition. Researchers have introduced new aggregation operators, ranking techniques, and decision-making models based on interval-valued intuitionistic fuzzy information. The interval-valued intuitionistic fuzzy best-worst method and its consistency analysis have notably advanced multi-criteria decision-making approaches [4]. Likewise, interval-valued intuitionistic fuzzy models have been effectively utilized in complex optimization challenges and energy management systems [3]. Additionally, quaternion representations and correlation measures for interval-valued intuitionistic fuzzy sets have broadened the theory's scope [9]. The algebraic exploration of interval-valued fuzzy structures has garnered significant interest. Various scholars have examined interval-valued fuzzy ideals, generalized interval-valued fuzzy ideals, spherical interval-valued fuzzy ideals, and interval-valued intuitionistic fuzzy sub semirings, uncovering intriguing structural properties of semirings and related algebraic systems [5–8]. The algebraic characteristics of interval-valued intuitionistic fuzzy numbers have also been thoroughly investigated, laying a solid theoretical groundwork for further advancements in fuzzy algebra [10]. Semirings represent a crucial category of algebraic structures with applications in automata theory, optimization, graph theory, coding theory, formal language theory, and computer science. The exploration of fuzzy ideals and intuitionistic fuzzy ideals in semirings offers a valuable framework for characterizing uncertainties in algebraic systems. Inspired by recent progress in interval-valued fuzzy algebra and semiring theory [5–10], it is logical to explore more robust forms of interval-valued intuitionistic fuzzy ideals that meet additional algebraic criteria. Recent advancements in fuzzy algebra have introduced several novel algebraic structures and generalized fuzzy ideals, such as HP-algebras and Q-fuzzy JU-algebras, which have enriched the study of fuzzy algebraic systems and their applications [11,12]. These developments encourage the extension of interval-valued intuitionistic fuzzy concepts to more robust ideal structures in semirings.

In this study, we present the concept of a strong interval-valued intuitionistic fuzzy ideal within a semiring. This innovative idea enhances the traditional interval-valued intuitionistic fuzzy ideal by adding extra conditions to the interval-valued membership and non-membership functions when considering semiring addition and multiplication. We identify several key properties of these ideals and offer illustrative examples. Additionally, we prove various structural results related to finite sums, finite products, arbitrary intersections, level subsets, homomorphisms, isomorphisms, and characterization theorems. These findings advance the theory of interval-valued intuitionistic fuzzy algebra and lay the groundwork for future investigations into prime ideals, maximal ideals, quotient semirings, and their applications in uncertainty modelling and intelligent decision-making.

2. Preliminaries

Throughout this paper, let Γdenote a semiring.

Definition 2.1

A semiring is an algebraic structure $(\Gamma, +, \cdot)$ consisting of a non-empty set Γ together with two binary operations $+$ and $\cdot$ satisfying the following axioms:

$(\Gamma, +)$ is a commutative monoid with identity element 0.

$(\Gamma, \cdot)$ is a semigroup.

Multiplication distributes over addition from both sides:

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma,$$
$$(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma.$$

$0\alpha = \alpha 0 = 0$for every $\alpha \in \Gamma$.

Definition 2.2

An interval-valued intuitionistic fuzzy set (IVIFS) Δon $\Gamma$is defined as $\Delta = \left\{\left(\alpha, \widetilde{M}_\Delta, (\alpha)\widetilde{N}_\Delta(\alpha)\right) : \alpha \in \Gamma\right\}$, where $\widetilde{M}_\Delta : \Gamma \to D[0,1]$, and $\widetilde{N}_\Delta : \Gamma \to D[0,1]$ are the interval-valued membership and non-membership functions satisfying $\sup\left(\widetilde{M}_\Delta(\alpha)\right) + \sup\left(\widetilde{N}_\Delta(\alpha)\right) \leq 1$, for every $\alpha \in \Gamma$[1,2].

Definition 2.3

For two intervals $\tilde{a} = [a^-, a^+]$, $\tilde{b} = [b^-, b^+] \in D[0,1]$, we write $\tilde{a} \leq \tilde{b}$ if $a^- \leq b^-$ and $a^+ \leq b^+$. Further, $\mathrm{rmin}(\tilde{a}, \tilde{b}) = [\min,(a^-, b^-)\min(a^+, b^+)]$, and $\mathrm{rmax}(\tilde{a}, \tilde{b}) = [\max,(a^-, b^-)\max(a^+, b^+)]$.

Definition 2.4

Let $\Delta = \left(\widetilde{M}_\Delta, \widetilde{N}_\Delta\right)$ be an interval-valued intuitionistic fuzzy subset of a semiring Γ. Then Δ is called a strong interval-valued intuitionistic fuzzy ideal of Γ if, for all $\alpha, \beta \in \Gamma$,

$$\widetilde{M}_\Delta(\alpha + \beta) \geq \mathrm{rmin}\{\widetilde{M}_\Delta, (\alpha)\widetilde{M}_\Delta(\beta)\},$$
$$\widetilde{M}_\Delta(\alpha\beta) \geq \mathrm{rmax}\{\widetilde{M}_\Delta, (\alpha)\widetilde{M}_\Delta(\beta)\},$$
$$\widetilde{N}_\Delta(\alpha + \beta) \leq \mathrm{rmax}\{\widetilde{N}_\Delta, (\alpha)\widetilde{N}_\Delta(\beta)\},$$

$\widetilde{N}_\Delta(\alpha\beta) \leq \mathrm{rmin}\{\widetilde{N}_\Delta, (\alpha)\widetilde{N}_\Delta(\beta)\}$, together with $\widetilde{M}_\Delta(\alpha + \alpha) = \widetilde{M}_\Delta(\alpha)$, and $\widetilde{N}_\Delta(\alpha + \alpha) = \widetilde{N}_\Delta(\alpha)$, for every $\alpha \in \Gamma$.

The initial sections introduce the fundamental ideas and notation that are consistently applied in the paper, serving as the groundwork for determining the characteristics and structural outcomes of strong interval-valued intuitionistic fuzzy ideals. This discussion is based on traditional fuzzy set theory, interval-valued intuitionistic fuzzy sets, and the latest developments in fuzzy algebra and semiring theory [1–14].

3. Strong interval-valued intuitionistic fuzzy ideal

Definition 3.1

Let Γ be a semiring and let $\Delta = \left(\widetilde{M}_\Delta, \widetilde{N}_\Delta\right)$ be an interval-valued intuitionistic fuzzy subset of Γ, where $\widetilde{M}_\Delta : \Gamma \to D[0,1]$ and $\widetilde{N}_\Delta : \Gamma \to D[0,1]$ denote the interval-valued membership and non-membership functions, respectively, satisfying

$$\sup\left(\widetilde{M}_\Delta(\alpha)\right) + \sup\left(\widetilde{N}_\Delta(\alpha)\right) \leq 1,\ \forall\, \alpha \in \Gamma.$$

Then Δ is called a strong interval-valued intuitionistic fuzzy ideal of Γ if, for all $\alpha, \beta \in \Gamma$,

$$\widetilde{M}_\Delta(\alpha+\beta) \geq \mathrm{rmin}\left\{\widetilde{M}_\Delta, (\alpha)\widetilde{M}_\Delta(\beta)\right\};$$
$$\widetilde{M}_\Delta(\alpha\beta) \geq \mathrm{rmax}\left\{\widetilde{M}_\Delta, (\alpha)\widetilde{M}_\Delta(\beta)\right\};$$
$$\widetilde{N}_\Delta(\alpha+\beta) \leq \mathrm{rmax}\left\{\widetilde{N}_\Delta, (\alpha)\widetilde{N}_\Delta(\beta)\right\};$$
$$\widetilde{N}_\Delta(\alpha\beta) \leq \mathrm{rmin}\left\{\widetilde{N}_\Delta, (\alpha)\widetilde{N}_\Delta(\beta)\right\};$$

$\widetilde{M}_\Delta(\alpha+\alpha) = \widetilde{M}_\Delta(\alpha),\ \widetilde{N}_\Delta(\alpha+\alpha) = \widetilde{N}_\Delta(\alpha)$, for every $\alpha \in \Gamma$.

Hence, $\Delta = \left(\widetilde{M}_\Delta, \widetilde{N}_\Delta\right)$ satisfying the above five conditions is called a strong interval-valued intuitionistic fuzzy ideal of the semiring Γ.

Example 3.2

Let $\Gamma = \{0, a, b\}$ be a semiring with the following addition and multiplication tables.

Addition

$+$	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

Multiplication

$\cdot$	0	a	b
0	0	0	0
a	0	a	b
b	0	b	b

Define an interval-valued intuitionistic fuzzy subset $\Delta = \left(\widetilde{M}_\Delta, \widetilde{N}_\Delta\right)$ of Γ by

x	$\widetilde{M}_\Delta(x)$	$\widetilde{N}_\Delta(x)$
0	$[1.00, 1.00]$	$[0.00, 0.00]$
a	$[0.70, 0.80]$	$[0.10, 0.20]$
b	$[0.40, 0.50]$	$[0.30, 0.40]$

Clearly, $\sup \widetilde{M}_\Delta(x) + \sup \widetilde{N}_\Delta(x) \leq 1$ for every $x \in \Gamma$. Indeed, $1+0=1,\ 0.8+0.2=1,\ 0.5+0.4=0.9 \leq 1$.
Now verify the defining conditions.

Addition, since $a+a=a,\ a+b=b,\ b+b=b$, we have

$$\widetilde{M}_\Delta(a+a) = [0.70, 0.80] = \mathrm{rmin}\{[0.70, 0.80], [0.70, 0.80]\},$$
$$\widetilde{M}_\Delta(a+b) = [0.40, 0.50] = \mathrm{rmin}\{[0.70, 0.80], [0.40, 0.50]\},$$

and similarly for all other sums. Also,

$$\widetilde{N}_\Delta(a+b) = [0.30, 0.40] = \mathrm{rmax}\{[0.10, 0.20], [0.30, 0.40]\}.$$

Multiplication, Since $a \cdot a = a,\ a \cdot b = b,\ b \cdot b = b$, we obtain

$$\widetilde{M}_\Delta(a^2) = [0.70, 0.80] = \mathrm{rmax}\{[0.70, 0.80], [0.70, 0.80]\},$$
$$\widetilde{M}_\Delta(ab) = [0.40, 0.50] \geq \mathrm{rmax}\{[0.70, 0.80], [0.40, 0.50]\},$$

and

$$\widetilde{N}_\Delta(ab) = [0.30, 0.40] \leq \mathrm{rmin}\{[0.10, 0.20], [0.30, 0.40]\}.$$

Idempotent condition, since $0+0=0,\ a+a=a,\ b+b=b$, we have $\widetilde{M}_\Delta(x+x) = \widetilde{M}_\Delta(x)$, and $\widetilde{N}_\Delta(x+x) = \widetilde{N}_\Delta(x)$, for every $x \in \Gamma$. Hence, all the defining conditions are satisfied. Therefore, $\Delta = \left(\widetilde{M}_\Delta, \widetilde{N}_\Delta\right)$ is a strong interval-valued intuitionistic fuzzy ideal of the semiring Γ.

Theorem 3.3

Let Γ be a semiring and let $\Delta_1 = \left(\widetilde{M}_1, \widetilde{N}_1\right)$ and $\Delta_2 = \left(\widetilde{M}_2, \widetilde{N}_2\right)$ be two strong interval-valued intuitionistic fuzzy ideals of Γ. Define $\Delta = \Delta_1 \cap \Delta_2 = \left(\widetilde{M}, \widetilde{N}\right)$, where $\widetilde{M}(\alpha) = \mathrm{rmin}\left\{\widetilde{M}_1, (\alpha)\widetilde{M}_2(\alpha)\right\}$, and $\widetilde{N}(\alpha) = \mathrm{rmax}\left\{\widetilde{N}_1, (\alpha)\widetilde{N}_2(\alpha)\right\}$, for every $\alpha \in \Gamma$. Then Δ is also a strong interval-valued intuitionistic fuzzy ideal of Γ.

Proof

Let $\alpha, \beta \in \Gamma$. Since Δ_1 and Δ_2 are strong interval-valued intuitionistic fuzzy ideals, $\widetilde{M}_i(\alpha+\beta) \geq \operatorname{rmin}\{\widetilde{M}_i, (\alpha)\widetilde{M}_i(\beta)\}$, $i = 1,2$. Hence,

$$\begin{aligned}\widetilde{M}(\alpha+\beta) &= \operatorname{rmin}\{\widetilde{M}_1, (\alpha+\beta)\widetilde{M}_2(\alpha+\beta)\}\\ &\geq \operatorname{rmin}\{\widetilde{M}, (\alpha)\widetilde{M}(\beta)\}.\end{aligned}$$

Similarly, $\widetilde{M}(\alpha\beta) \geq \operatorname{rmax}\{\widetilde{M}, (\alpha)\widetilde{M}(\beta)\}$. Also, $\widetilde{N}(\alpha+\beta) \leq \operatorname{rmax}\{\widetilde{N}, (\alpha)\widetilde{N}(\beta)\}$, and $\widetilde{N}(\alpha\beta) \leq \operatorname{rmin}\{\widetilde{N}, (\alpha)\widetilde{N}(\beta)\}$. Finally, since each $\Delta_i$satisfies the idempotent property, $\widetilde{M}_i(\alpha+\alpha) = \widetilde{M}_i(\alpha)$, $\widetilde{N}_i(\alpha+\alpha) = \widetilde{N}_i(\alpha)$, we obtain

$$\begin{aligned}\widetilde{M}(\alpha+\alpha) &= \operatorname{rmin}\{\widetilde{M}_1, (\alpha+\alpha)\widetilde{M}_2(\alpha+\alpha)\}\\ &= \operatorname{rmin}\{\widetilde{M}_1, (\alpha)\widetilde{M}_2(\alpha)\} = \widetilde{M}(\alpha),\end{aligned}$$

and

$$\begin{aligned}\widetilde{N}(\alpha+\alpha) &= \operatorname{rmax}\{\widetilde{N}_1, (\alpha+\alpha)\widetilde{N}_2(\alpha+\alpha)\}\\ &= \operatorname{rmax}\{\widetilde{N}_1, (\alpha)\widetilde{N}_2(\alpha)\} = \widetilde{N}(\alpha).\end{aligned}$$

Therefore, Δ satisfies all the defining conditions of a strong interval-valued intuitionistic fuzzy ideal of Γ. Hence, $\Delta_1 \cap \Delta_2$ is a strong interval-valued intuitionistic fuzzy ideal of Γ

Theorem 3.4

Let Γ be a semiring and let $\Delta = (\widetilde{M}_\Delta, \widetilde{N}_\Delta)$ be a strong interval-valued intuitionistic fuzzy ideal of Γ. If $e \in \Gamma$ is an idempotent element, that is, $e + e = e$, then $\widetilde{M}_\Delta(e+e) = \widetilde{M}_\Delta(e)$
And $\widetilde{N}_\Delta(e+e) = \widetilde{N}_\Delta(e)$.

Proof

Since e is idempotent, $e + e = e$. Therefore, $\widetilde{M}_\Delta(e+e) = \widetilde{M}_\Delta(e)$, and $\widetilde{N}_\Delta(e+e) = \widetilde{N}_\Delta(e)$. Moreover, by the idempotent condition in the definition of a strong interval-valued intuitionistic fuzzy ideal, $\widetilde{M}_\Delta(e+e) = \widetilde{M}_\Delta(e)$, and $\widetilde{N}_\Delta(e+e) = \widetilde{N}_\Delta(e)$. Hence, every idempotent element of the semiring preserves its interval-valued intuitionistic fuzzy membership and non-membership values under addition. Therefore, $\widetilde{M}_\Delta(e+e) = \widetilde{M}_\Delta(e)$ and $\widetilde{N}_\Delta(e+e) = \widetilde{N}_\Delta(e)$.

Lemma 3.5

Let Γ be a semiring and let $\Delta = (\widetilde{M}_\Delta, \widetilde{N}_\Delta)$ be a strong interval-valued intuitionistic fuzzy ideal of Γ. Then, for any finite collection of elements $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Gamma$ $(n \geq 2)$, the following hold:

$$\widetilde{M}_\Delta(\alpha_1+\alpha_2+\cdots+\alpha_n) \geq \operatorname{rmin}\{\widetilde{M}_\Delta, (\alpha_1), \widetilde{M}_\Delta, (\alpha_2) \ldots \widetilde{M}_\Delta(\alpha_n)\},$$

and

$$\widetilde{N}_\Delta(\alpha_1+\alpha_2+\cdots+\alpha_n) \leq \operatorname{rmax}\{\widetilde{N}_\Delta, (\alpha_1), \widetilde{N}_\Delta, (\alpha_2) \ldots \widetilde{N}_\Delta(\alpha_n)\}.$$

Proof

We prove the result by mathematical induction on n. For $n = 2$, the result follows directly from the definition of a strong interval-valued intuitionistic fuzzy ideal:

$$\widetilde{M}_\Delta(\alpha_1+\alpha_2) \geq \operatorname{rmin}\{\widetilde{M}_\Delta, (\alpha_1)\widetilde{M}_\Delta(\alpha_2)\},$$

and

$$\widetilde{N}_\Delta(\alpha_1+\alpha_2) \leq \operatorname{rmax}\{\widetilde{N}_\Delta, (\alpha_1)\widetilde{N}_\Delta(\alpha_2)\}.$$

Assume the statement is true for $n = k$. Then

$$\widetilde{M}_\Delta\left(\sum_{i=1}^{k}\alpha_i\right) \geq \operatorname{rmin}\{\widetilde{M}_\Delta, (\alpha_1), \ldots \widetilde{M}_\Delta(\alpha_k)\}.$$

Now consider $n = k + 1$. Using the defining property,

$$\begin{aligned}\widetilde{M}_\Delta\left(\sum_{i=1}^{k+1}\alpha_i\right) &= \widetilde{M}_\Delta\left(\left(\sum_{i=1}^{k}\alpha_i\right)+\alpha_{k+1}\right)\\ &\geq \operatorname{rmin}\{\widetilde{M}_\Delta, (\textstyle\sum_{i=1}^{k}\alpha_i)\widetilde{M}_\Delta(\alpha_{k+1})\}\\ &\geq \operatorname{rmin}\{\widetilde{M}_\Delta, (\alpha_1), \ldots, \widetilde{M}_\Delta(\alpha_k)\widetilde{M}_\Delta(\alpha_{k+1})\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\widetilde{N}_\Delta\left(\sum_{i=1}^{k+1}\alpha_i\right) &\leq \operatorname{rmax}\{\widetilde{N}_\Delta, (\textstyle\sum_{i=1}^{k}\alpha_i)\widetilde{N}_\Delta(\alpha_{k+1})\}\\ &\leq \operatorname{rmax}\{\widetilde{N}_\Delta, (\alpha_1), \ldots, \widetilde{N}_\Delta(\alpha_k)\widetilde{N}_\Delta(\alpha_{k+1})\}.\end{aligned}$$

Hence, the result holds for $k + 1$. By induction, it is true for every $n \geq 2$. Therefore,

$$\widetilde{M}_{\Delta}\left(\sum_{i=1}^{n}\alpha_i\right) \geq \mathrm{rmin}_{1\leq i\leq n}\widetilde{M}_{\Delta}(\alpha_i),$$
$$\widetilde{N}_{\Delta}\left(\sum_{i=1}^{n}\alpha_i\right) \leq \mathrm{rmax}_{1\leq i\leq n}\widetilde{N}_{\Delta}(\alpha_i).$$

Theorem 3.6

Let Γ_1 and Γ_2 be semirings, and let $f: \Gamma_1 \longrightarrow \Gamma_2$ be a surjective semiring homomorphism. Suppose that $\Delta = (\widetilde{M}_{\Delta}, \widetilde{N}_{\Delta})$ is a strong interval-valued intuitionistic fuzzy ideal of Γ_2. Define the inverse image of Δ under f by $f^{-1}(\Delta) = (\widetilde{M}_{f^{-1}(\Delta)}, \widetilde{N}_{f^{-1}(\Delta)})$, where $\widetilde{M}_{f^{-1}(\Delta)}(x) = \widetilde{M}_{\Delta}(f(x))$, and $\widetilde{N}_{f^{-1}(\Delta)}(x) = \widetilde{N}_{\Delta}(f(x))$, $\forall x \in \Gamma_1$. Then $f^{-1}(\Delta)$ is a strong interval-valued intuitionistic fuzzy ideal of Γ_1.

Proof

Let $x, y \in \Gamma_1$. Since f is a semiring homomorphism, $f(x+y) = f(x) + f(y)$, And $f(xy) = f(x)f(y)$. Hence,

$$\begin{aligned}\widetilde{M}_{f^{-1}(\Delta)}(x+y) &= \widetilde{M}_{\Delta}(f(x+y)) \\ &= \widetilde{M}_{\Delta}(f(x)+f(y)) \\ &\geq \mathrm{rmin}\{\widetilde{M}_{\Delta}, (f(x))\widetilde{M}_{\Delta}(f(y))\} \\ &= \mathrm{rmin}\{\widetilde{M}_{f^{-1}(\Delta)}, (x)\widetilde{M}_{f^{-1}(\Delta)}(y)\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\widetilde{M}_{f^{-1}(\Delta)}(xy) &= \widetilde{M}_{\Delta}(f(xy)) \\ &= \widetilde{M}_{\Delta}(f(x)f(y)) \\ &\geq \mathrm{rmax}\{\widetilde{M}_{f^{-1}(\Delta)}, (x)\widetilde{M}_{f^{-1}(\Delta)}(y)\}.\end{aligned}$$

Also,

$$\begin{aligned}\widetilde{N}_{f^{-1}(\Delta)}(x+y) &= \widetilde{N}_{\Delta}(f(x+y)) \\ &\leq \mathrm{rmax}\{\widetilde{N}_{f^{-1}(\Delta)}, (x)\widetilde{N}_{f^{-1}(\Delta)}(y)\},\end{aligned}$$

and

$$\begin{aligned}\widetilde{N}_{f^{-1}(\Delta)}(xy) &= \widetilde{N}_{\Delta}(f(xy)) \\ &\leq \mathrm{rmin}\{\widetilde{N}_{f^{-1}(\Delta)}, (x)\widetilde{N}_{f^{-1}(\Delta)}(y)\}.\end{aligned}$$

Finally,

$$\begin{aligned}\widetilde{M}_{f^{-1}(\Delta)}(x+x) &= \widetilde{M}_{\Delta}(f(x+x)) \\ &= \widetilde{M}_{\Delta}(f(x)+f(x)) \\ &= \widetilde{M}_{\Delta}(f(x)) \\ &= \widetilde{M}_{f^{-1}(\Delta)}(x),\end{aligned}$$

and similarly, $\widetilde{N}_{f^{-1}(\Delta)}(x+x) = \widetilde{N}_{f^{-1}(\Delta)}(x)$. Therefore, $f^{-1}(\Delta)$ satisfies all the defining properties of a strong interval-valued intuitionistic fuzzy ideal of Γ_1.

Hence, $f^{-1}(\Delta)$ is a strong interval-valued intuitionistic fuzzy ideal of Γ_1.

Theorem 3.7

Let Γ be a semiring and let $\Delta = (\widetilde{M}_{\Delta}, \widetilde{N}_{\Delta})$ be a strong interval-valued intuitionistic fuzzy ideal of Γ. For fixed intervals $\tilde{\lambda}, \tilde{\eta} \in D[0,1]$, define the level subsets $U(\tilde{\lambda}) = \{\alpha \in \Gamma: \widetilde{M}_{\Delta}(\alpha) \geq \tilde{\lambda}\}$, and $L(\tilde{\eta}) = \{\alpha \in \Gamma: \widetilde{N}_{\Delta}(\alpha) \leq \tilde{\eta}\}$. If $U(\tilde{\lambda})$and $L(\tilde{\eta})$are non-empty, then they are ideals of the semiring Γ.

Proof

Let $x, y \in U(\tilde{\lambda})$. Then $\widetilde{M}_{\Delta}(x) \geq \tilde{\lambda}$, $\widetilde{M}_{\Delta}(y) \geq \tilde{\lambda}$. Since Δ is a strong interval-valued intuitionistic fuzzy ideal, $\widetilde{M}_{\Delta}(x+y) \geq \mathrm{rmin}\{\widetilde{M}_{\Delta}, (x)\widetilde{M}_{\Delta}(y)\} \geq \tilde{\lambda}$. Hence, $x + y \in U(\tilde{\lambda})$. Also, $\widetilde{M}_{\Delta}(xy) \geq \mathrm{rmax}\{\widetilde{M}_{\Delta}, (x)\widetilde{M}_{\Delta}(y)\} \geq \tilde{\lambda}$, which implies $xy \in U(\tilde{\lambda})$. Therefore, $U(\tilde{\lambda})$ is closed under addition and multiplication, so it is an ideal of Γ. Similarly, let $x, y \in L(\tilde{\eta})$. Then $\widetilde{N}_{\Delta}(x) \leq \tilde{\eta}$, $\widetilde{N}_{\Delta}(y) \leq \tilde{\eta}$. Using the defining properties, $\widetilde{N}_{\Delta}(x+y) \leq \mathrm{rmax}\{\widetilde{N}_{\Delta}, (x)\widetilde{N}_{\Delta}(y)\} \leq \tilde{\eta}$, and $\widetilde{N}_{\Delta}(xy) \leq \mathrm{rmin}\{\widetilde{N}_{\Delta}, (x)\widetilde{N}_{\Delta}(y)\} \leq \tilde{\eta}$. Hence, $x + y$, $xy \in L(\tilde{\eta})$, showing that $L(\tilde{\eta})$ is also an ideal of Γ. Therefore, every non-empty membership level subset and every non-empty non-membership level subset of a strong interval-valued intuitionistic fuzzy ideal are ideals of the semiring Γ. $U(\tilde{\lambda})$ and $L(\tilde{\eta})$ are ideals of Γ.

Theorem 3.8

Let Γ be a semiring and let $\Delta = (\widetilde{M}_\Delta, \widetilde{N}_\Delta)$ be a strong interval-valued intuitionistic fuzzy ideal of Γ. For any intervals $\tilde{\lambda}, \tilde{\eta} \in D[0,1]$, define $S(\tilde{\lambda}, \tilde{\eta}) = \{\alpha \in \Gamma : \widetilde{M}_\Delta(\alpha) \geq \tilde{\lambda} \text{ and } \widetilde{N}_\Delta(\alpha) \leq \tilde{\eta}\}$. If $S(\tilde{\lambda}, \tilde{\eta}) \neq \emptyset$, then $S(\tilde{\lambda}, \tilde{\eta})$is an ideal of Γ.

Proof

Let $x, y \in S(\tilde{\lambda}, \tilde{\eta})$. Then $\widetilde{M}_\Delta(x) \geq \tilde{\lambda}$, $\widetilde{M}_\Delta(y) \geq \tilde{\lambda}$, and $\widetilde{N}_\Delta(x) \leq \tilde{\eta}$, $\widetilde{N}_\Delta(y) \leq \tilde{\eta}$. Since Δ is a strong interval-valued intuitionistic fuzzy ideal, $\widetilde{M}_\Delta(x+y) \geq \text{rmin}\{\widetilde{M}_\Delta, (x)\widetilde{M}_\Delta(y)\} \geq \tilde{\lambda}$, and $\widetilde{N}_\Delta(x+y) \leq \text{rmax}\{\widetilde{N}_\Delta, (x)\widetilde{N}_\Delta(y)\} \leq \tilde{\eta}$. Hence, $x + y \in S(\tilde{\lambda}, \tilde{\eta})$. Similarly, $\widetilde{M}_\Delta(xy) \geq \text{rmax}\{\widetilde{M}_\Delta, (x)\widetilde{M}_\Delta(y)\} \geq \tilde{\lambda}$, and $\widetilde{N}_\Delta(xy) \leq \text{rmin}\{\widetilde{N}_\Delta, (x)\widetilde{N}_\Delta(y)\} \leq \tilde{\eta}$. Therefore, $xy \in S(\tilde{\lambda}, \tilde{\eta})$. Thus, $S(\tilde{\lambda}, \tilde{\eta})$ is closed under both addition and multiplication. Hence, $S(\tilde{\lambda}, \tilde{\eta})$ is an ideal of Γ. Therefore, $S(\tilde{\lambda}, \tilde{\eta})$ is an ideal of Γ

Corollary 3.9

For every pair of intervals $(\tilde{\lambda}, \tilde{\eta})$, the family of non-empty sets $\{S, (\tilde{\lambda}, \tilde{\eta}) : \tilde{\lambda}\tilde{\eta} \in D[0,1]\}$ forms a nested system of ideals of Γ.

Theorem 3.10

Let Γ be a semiring and let $\Delta = (\widetilde{M}_\Delta, \widetilde{N}_\Delta)$ be a strong interval-valued intuitionistic fuzzy ideal of Γ. If $\alpha, \beta \in \Gamma$ satisfy $\widetilde{M}_\Delta(\alpha) = \widetilde{M}_\Delta(\beta)$ and $\widetilde{N}_\Delta(\alpha) = \widetilde{N}_\Delta(\beta)$, then $\widetilde{M}_\Delta(\alpha\beta) \geq \widetilde{M}_\Delta(\alpha)$, and $\widetilde{N}_\Delta(\alpha\beta) \leq \widetilde{N}_\Delta(\alpha)$.

Proof

Since $\widetilde{M}_\Delta(\alpha) = \widetilde{M}_\Delta(\beta)$, we have $\text{rmax}\{\widetilde{M}_\Delta, (\alpha)\widetilde{M}_\Delta(\beta)\} = \widetilde{M}_\Delta(\alpha)$. By the definition of a strong interval-valued intuitionistic fuzzy ideal, $\widetilde{M}_\Delta(\alpha\beta) \geq \text{rmax}\{\widetilde{M}_\Delta, (\alpha)\widetilde{M}_\Delta(\beta)\}$. Hence, $\widetilde{M}_\Delta(\alpha\beta) \geq \widetilde{M}_\Delta(\alpha)$. Similarly, $\widetilde{N}_\Delta(\alpha) = \widetilde{N}_\Delta(\beta)$, implies $\text{rmin}\{\widetilde{N}_\Delta, (\alpha)\widetilde{N}_\Delta(\beta)\} = \widetilde{N}_\Delta(\alpha)$. Again, by definition, $\widetilde{N}_\Delta(\alpha\beta) \leq \text{rmin}\{\widetilde{N}_\Delta, (\alpha)\widetilde{N}_\Delta(\beta)\}$, which yields $\widetilde{N}_\Delta(\alpha\beta) \leq \widetilde{N}_\Delta(\alpha)$. Therefore, $\widetilde{M}_\Delta(\alpha\beta) \geq \widetilde{M}_\Delta(\alpha)$ and $\widetilde{N}_\Delta(\alpha\beta) \leq \widetilde{N}_\Delta(\alpha)$

Theorem 3.11

Let Γ be a semiring and let $\Delta = (\widetilde{M}_\Delta, \widetilde{N}_\Delta)$ be a strong interval-valued intuitionistic fuzzy ideal of Γ. Then, for any finite collection of elements $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Gamma$, $n \geq 2$, the following statements hold: $\widetilde{M}_\Delta(\prod_{i=1}^{n} \alpha_i) \geq \text{rmax}\{\widetilde{M}_\Delta, (\alpha_1), \widetilde{M}_\Delta, (\alpha_2) \ldots \widetilde{M}_\Delta(\alpha_n)\}$, and $\widetilde{N}_\Delta(\prod_{i=1}^{n} \alpha_i) \leq \text{rmin}\{\widetilde{N}_\Delta, (\alpha_1), \widetilde{N}_\Delta, (\alpha_2) \ldots \widetilde{N}_\Delta(\alpha_n)\}$.

Proof

We prove the theorem by induction on n. For $n = 2$, the result follows directly from the definition of a strong interval-valued intuitionistic fuzzy ideal: $\widetilde{M}_\Delta(\alpha_1\alpha_2) \geq \text{rmax}\{\widetilde{M}_\Delta, (\alpha_1)\widetilde{M}_\Delta(\alpha_2)\}$, and $\widetilde{N}_\Delta(\alpha_1\alpha_2) \leq \text{rmin}\{\widetilde{N}_\Delta, (\alpha_1)\widetilde{N}_\Delta(\alpha_2)\}$. Assume the theorem is true for $n = k$. Then $\widetilde{M}_\Delta(\prod_{i=1}^{k} \alpha_i) \geq \text{rmax}\{\widetilde{M}_\Delta, (\alpha_1), \ldots \widetilde{M}_\Delta(\alpha_k)\}$, and $\widetilde{N}_\Delta(\prod_{i=1}^{k} \alpha_i) \leq \text{rmin}\{\widetilde{N}_\Delta, (\alpha_1), \ldots \widetilde{N}_\Delta(\alpha_k)\}$. Now consider the product of $k+1$elements: $\prod_{i=1}^{k+1} \alpha_i = (\prod_{i=1}^{k} \alpha_i)\alpha_{k+1}$. Using the defining property of Δ,

$$\begin{aligned}\widetilde{M}_\Delta\left(\prod_{i=1}^{k+1} \alpha_i\right) &= \widetilde{M}_\Delta\left(\left(\prod_{i=1}^{k} \alpha_i\right)\alpha_{k+1}\right) \\ &\geq \text{rmax}\{\widetilde{M}_\Delta, (\textstyle\prod_{i=1}^{k} \alpha_i)\widetilde{M}_\Delta(\alpha_{k+1})\} \\ &\geq \text{rmax}\{\widetilde{M}_\Delta, (\alpha_1), \ldots, \widetilde{M}_\Delta(\alpha_k)\widetilde{M}_\Delta(\alpha_{k+1})\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\widetilde{N}_\Delta\left(\prod_{i=1}^{k+1} \alpha_i\right) &= \widetilde{N}_\Delta\left(\left(\prod_{i=1}^{k} \alpha_i\right)\alpha_{k+1}\right) \\ &\leq \text{rmin}\{\widetilde{N}_\Delta, (\textstyle\prod_{i=1}^{k} \alpha_i)\widetilde{N}_\Delta(\alpha_{k+1})\} \\ &\leq \text{rmin}\{\widetilde{N}_\Delta, (\alpha_1), \ldots, \widetilde{N}_\Delta(\alpha_k)\widetilde{N}_\Delta(\alpha_{k+1})\}.\end{aligned}$$

Thus, the theorem holds for $k + 1$. Hence, by mathematical induction, it holds for every $n \geq 2$. Therefore,

$$\widetilde{M}_\Delta\left(\prod_{i=1}^{n} \alpha_i\right) \geq \text{rmax}_{1 \leq i \leq n} \widetilde{M}_\Delta(\alpha_i),$$

$$\widetilde{N}_\Delta\left(\prod_{i=1}^{n} \alpha_i\right) \leq \text{rmin}_{1 \leq i \leq n} \widetilde{N}_\Delta(\alpha_i).$$

Theorem 3.12

Let Γ be a semiring and let $\Delta = (\widetilde{M}_\Delta, \widetilde{N}_\Delta)$ be a strong interval-valued intuitionistic fuzzy ideal of Γ. If there exist intervals $\tilde{\lambda}, \tilde{\eta} \in D[0,1]$ such that $\widetilde{M}_\Delta(\alpha) = \tilde{\lambda}$ and $\widetilde{N}_\Delta(\alpha) = \tilde{\eta}$, $\forall\, \alpha \in \Gamma$, then Δ is a strong interval-valued intuitionistic fuzzy ideal of Γ.

Proof

Since $\widetilde{M}_\Delta$ and $\widetilde{N}_\Delta$ are constant functions on Γ, for any $\alpha, \beta \in \Gamma$, $\widetilde{M}_\Delta(\alpha+\beta) = \tilde{\lambda} = \text{rmin}\{\tilde{\lambda}, \tilde{\lambda}\} = \text{rmin}\{\widetilde{M}_\Delta, (\alpha)\widetilde{M}_\Delta(\beta)\}$. Similarly, $\widetilde{M}_\Delta(\alpha\beta) = \tilde{\lambda} = \text{rmax}\{\tilde{\lambda}, \tilde{\lambda}\} = \text{rmax}\{\widetilde{M}_\Delta, (\alpha)\widetilde{M}_\Delta(\beta)\}$. Also, $\widetilde{N}_\Delta(\alpha+\beta) = \tilde{\eta} = \text{rmax}\{\tilde{\eta}, \tilde{\eta}\} = \text{rmax}\{\widetilde{N}_\Delta, (\alpha)\widetilde{N}_\Delta(\beta)\}$, and $\widetilde{N}_\Delta(\alpha\beta) = \tilde{\eta} = \text{rmin}\{\tilde{\eta}, \tilde{\eta}\} = \text{rmin}\{\widetilde{N}_\Delta, (\alpha)\widetilde{N}_\Delta(\beta)\}$. Finally, $\widetilde{M}_\Delta(\alpha+\alpha) = \tilde{\lambda} = \widetilde{M}_\Delta(\alpha)$, and $\widetilde{N}_\Delta(\alpha+\alpha) = \tilde{\eta} = \widetilde{N}_\Delta(\alpha)$. Hence, all the defining conditions of a strong interval-valued intuitionistic fuzzy ideal are satisfied. Therefore, $\Delta = (\widetilde{M}_\Delta, \widetilde{N}_\Delta)$ is a strong interval-valued intuitionistic fuzzy ideal of Γ.

Theorem 3.13

Let Γ be a semiring and let $\Delta_1 = (\widetilde{M}_1, \widetilde{N}_1)$ and $\Delta_2 = (\widetilde{M}_2, \widetilde{N}_2)$ be two strong interval-valued intuitionistic fuzzy ideals of Γ such that $\widetilde{M}_1(\alpha) \le \widetilde{M}_2(\alpha)$ and $\widetilde{N}_1(\alpha) \ge \widetilde{N}_2(\alpha)$, $\forall\, \alpha \in \Gamma$. Then, for every $\alpha, \beta \in \Gamma$,

$$\widetilde{M}_1(\alpha+\beta) \le \widetilde{M}_2(\alpha+\beta),$$
$$\widetilde{M}_1(\alpha\beta) \le \widetilde{M}_2(\alpha\beta),$$
$$\widetilde{N}_1(\alpha+\beta) \ge \widetilde{N}_2(\alpha+\beta),$$

and

$$\widetilde{N}_1(\alpha\beta) \ge \widetilde{N}_2(\alpha\beta).$$

Proof

Since $\widetilde{M}_1(\alpha) \le \widetilde{M}_2(\alpha)$ and $\widetilde{M}_1(\beta) \le \widetilde{M}_2(\beta)$, the monotonicity of the interval minimum implies, $\text{rmin}\{\widetilde{M}_1, (\alpha)\widetilde{M}_1(\beta)\} \le \text{rmin}\{\widetilde{M}_2, (\alpha)\widetilde{M}_2(\beta)\}$. Since both Δ_1 and Δ_2 are strong interval-valued intuitionistic fuzzy ideals, $\widetilde{M}_1(\alpha+\beta) \ge \text{rmin}\{\widetilde{M}_1, (\alpha)\widetilde{M}_1(\beta)\}$, and $\widetilde{M}_2(\alpha+\beta) \ge \text{rmin}\{\widetilde{M}_2, (\alpha)\widetilde{M}_2(\beta)\}$. Hence, $\widetilde{M}_1(\alpha+\beta) \le \widetilde{M}_2(\alpha+\beta)$. Similarly, $\widetilde{M}_1(\alpha\beta) \le \widetilde{M}_2(\alpha\beta)$. Also, $\widetilde{N}_1(\alpha) \ge \widetilde{N}_2(\alpha)$, $\widetilde{N}_1(\beta) \ge \widetilde{N}_2(\beta)$, imply $\text{rmax}\{\widetilde{N}_1, (\alpha)\widetilde{N}_1(\beta)\} \ge \text{rmax}\{\widetilde{N}_2, (\alpha)\widetilde{N}_2(\beta)\}$. Using the defining properties again, $\widetilde{N}_1(\alpha+\beta) \ge \widetilde{N}_2(\alpha+\beta)$, and $\widetilde{N}_1(\alpha\beta) \ge \widetilde{N}_2(\alpha\beta)$.

Therefore,

$$\widetilde{M}_1(\alpha+\beta) \le \widetilde{M}_2(\alpha+\beta),$$
$$\widetilde{M}_1(\alpha\beta) \le \widetilde{M}_2(\alpha\beta),$$
$$\widetilde{N}_1(\alpha+\beta) \ge \widetilde{N}_2(\alpha+\beta),$$
$$\widetilde{N}_1(\alpha\beta) \ge \widetilde{N}_2(\alpha\beta).$$

Hence, the ordering between two strong interval-valued intuitionistic fuzzy ideals is preserved under the semiring operations.

Theorem 3.14

Let Γ be a semiring and let $\{\Delta_i = (\widetilde{M}_i, \widetilde{N}_i) : i \in I\}$ be a family of strong interval-valued intuitionistic fuzzy ideals of Γ, where I is an arbitrary index set. Define $\Delta = \bigcap_{i\in I} \Delta_i = (\widetilde{M}, \widetilde{N})$, where $\widetilde{M}(\alpha) = \text{rinf}\{\widetilde{M}_i(\alpha) : i \in I\}$, and $\widetilde{N}(\alpha) = \text{rsup}\{\widetilde{N}_i(\alpha) : i \in I\}$, for every $\alpha \in \Gamma$. Then Δ is a strong interval-valued intuitionistic fuzzy ideal of Γ.

Proof

Let $\alpha, \beta \in \Gamma$. Since each Δ_i is a strong interval-valued intuitionistic fuzzy ideal, $\widetilde{M}_i(\alpha+\beta) \ge \text{rmin}\{\widetilde{M}_i, (\alpha)\widetilde{M}_i(\beta)\}$, $\forall i \in I$. Taking the interval infimum over all $i \in I$,

$$\begin{aligned}\widetilde{M}(\alpha+\beta) &= \text{rinf}\{\widetilde{M}_i(\alpha+\beta) : i \in I\}\\ &\ge \text{rmin}\{\text{rinf}\,\widetilde{M}_i, (\alpha)\,\text{rinf}\,\widetilde{M}_i\,(\beta)\}\\ &= \text{rmin}\{\widetilde{M}, (\alpha)\widetilde{M}(\beta)\}.\end{aligned}$$

Similarly, $\widetilde{M}(\alpha\beta) \ge \text{rmax}\{\widetilde{M}, (\alpha)\widetilde{M}(\beta)\}$. Also, $\widetilde{N}_i(\alpha+\beta) \le \text{rmax}\{\widetilde{N}_i, (\alpha)\widetilde{N}_i(\beta)\}$, $\forall i \in I$. Taking the interval supremum, $\widetilde{N}(\alpha+\beta) \le \text{rmax}\{\widetilde{N}, (\alpha)\widetilde{N}(\beta)\}$. Likewise, $\widetilde{N}(\alpha\beta) \le \text{rmin}\{\widetilde{N}, (\alpha)\widetilde{N}(\beta)\}$. Finally, $\widetilde{M}_i(\alpha+\alpha) = \widetilde{M}_i(\alpha)$, $\widetilde{N}_i(\alpha+\alpha) = \widetilde{N}_i(\alpha)$, $\forall i \in I$. Hence, $\widetilde{M}(\alpha+\alpha) = \widetilde{M}(\alpha)$, and $\widetilde{N}(\alpha+\alpha) = \widetilde{N}(\alpha)$. Therefore, Δ satisfies all the defining properties of a strong interval-valued intuitionistic fuzzy ideal of Γ. Thus, $\bigcap_{i\in I} \Delta_i$ is a strong interval-valued intuitionistic fuzzy ideal of Γ.

Theorem 3.15

Let Γ be a semiring and let $\Delta = (\widetilde{M}_\Delta, \widetilde{N}_\Delta)$ be an interval-valued intuitionistic fuzzy subset of Γ. For every interval $\tilde{\lambda}, \tilde{\eta} \in D[0,1]$, define the level subset $S(\tilde{\lambda}, \tilde{\eta}) = \{\alpha, \in \Gamma : \widetilde{M}_\Delta(\alpha) \ge \tilde{\lambda}\ \widetilde{N}_\Delta(\alpha) \le \tilde{\eta}\}$. Then Δ is a strong interval-valued intuitionistic fuzzy ideal of Γ if and only if every non-empty level subset $S(\tilde{\lambda}, \tilde{\eta})$ is an ideal of Γ and satisfies $\alpha \in S(\tilde{\lambda}, \tilde{\eta}) \Rightarrow \alpha + \alpha \in S(\tilde{\lambda}, \tilde{\eta})$.

Proof

We prove both implications. **(⇒)** Assume that Δ is a strong interval-valued intuitionistic fuzzy ideal. Let $x, y \in S(\tilde{\lambda}, \tilde{\eta})$. Then $\widetilde{M}_\Delta(x), \widetilde{M}_\Delta(y) \ge \tilde{\lambda}$, and $\widetilde{N}_\Delta(x), \widetilde{N}_\Delta(y) \le \tilde{\eta}$. By the defining properties, $\widetilde{M}_\Delta(x+y) \ge$

$\text{rmin}\{\widetilde{M}_{\Delta},(x)\widetilde{M}_{\Delta}(y)\} \geq \tilde{\lambda}$, and $\widetilde{N}_{\Delta}(x+y) \leq \text{rmax}\{\widetilde{N}_{\Delta},(x)\widetilde{N}_{\Delta}(y)\} \leq \tilde{\eta}$. Hence, $x+y \in S(\tilde{\lambda},\tilde{\eta})$. Similarly, $xy \in S(\tilde{\lambda},\tilde{\eta})$. Moreover, $\widetilde{M}_{\Delta}(x+x) = \widetilde{M}_{\Delta}(x)$, and $\widetilde{N}_{\Delta}(x+x) = \widetilde{N}_{\Delta}(x)$, which imply $x+x \in S(\tilde{\lambda},\tilde{\eta})$. Therefore every non-empty level subset is an ideal satisfying the idempotent condition.

(⇐) Conversely, assume every non-empty level subset $S(\tilde{\lambda},\tilde{\eta})$ is an ideal and satisfies $x \in S(\tilde{\lambda},\tilde{\eta}) \Longrightarrow x+x \in S(\tilde{\lambda},\tilde{\eta})$. Choose arbitrary $x,y \in \Gamma$. Set $\tilde{\lambda} = \text{rmin}\{\widetilde{M}_{\Delta},(x)\widetilde{M}_{\Delta}(y)\}$, and $\tilde{\eta} = \text{rmax}\{\widetilde{N}_{\Delta},(x)\widetilde{N}_{\Delta}(y)\}$. Then $x,y \in S(\tilde{\lambda},\tilde{\eta})$. Since $S(\tilde{\lambda},\tilde{\eta})$is an ideal, $x+y,\ xy \in S(\tilde{\lambda},\tilde{\eta})$, which gives

$$\widetilde{M}_{\Delta}(x+y) \geq \text{rmin}\{\widetilde{M}_{\Delta},(x)\widetilde{M}_{\Delta}(y)\},$$
$$\widetilde{M}_{\Delta}(xy) \geq \text{rmax}\{\widetilde{M}_{\Delta},(x)\widetilde{M}_{\Delta}(y)\},$$
$$\widetilde{N}_{\Delta}(x+y) \leq \text{rmax}\{\widetilde{N}_{\Delta},(x)\widetilde{N}_{\Delta}(y)\},$$

And $\widetilde{N}_{\Delta}(xy) \leq \text{rmin}\{\widetilde{N}_{\Delta},(x)\widetilde{N}_{\Delta}(y)\}$. Finally, the closure under $x+x$yields $\widetilde{M}_{\Delta}(x+x) = \widetilde{M}_{\Delta}(x)$, and $\widetilde{N}_{\Delta}(x+x) = \widetilde{N}_{\Delta}(x)$. Thus, *Δ*satisfies all the defining conditions of a strong interval-valued intuitionistic fuzzy ideal. Hence, $\Delta \text{ is a strong interval}-\text{valued intuitionistic fuzzy ideal of } \Gamma \text{ every non-empty level subset } S(\tilde{\lambda},\tilde{\eta}) \text{ is an ideal of } \Gamma.$

Theorem 3.16

Let Γ_1 and Γ_2 be semirings, and let $f\colon \Gamma_1 \to \Gamma_2$ be a semiring isomorphism. Suppose that $\Delta = (\widetilde{M}_{\Delta},\widetilde{N}_{\Delta})$ is a strong interval-valued intuitionistic fuzzy ideal of Γ_1. Define the image of Δ under f by $f(\Delta) = (\widetilde{M}_{f(\Delta)},\widetilde{N}_{f(\Delta)})$, where $\widetilde{M}_{f(\Delta)}(y) = \widetilde{M}_{\Delta}\left(f^{-1}(y)\right)$, and $\widetilde{N}_{f(\Delta)}(y) = \widetilde{N}_{\Delta}\left(f^{-1}(y)\right)$, for every $y \in \Gamma_2$. Then $f(\Delta)$ is a strong interval-valued intuitionistic fuzzy ideal of Γ_2.

Proof

Let $y_1,y_2 \in \Gamma_2$. Since f is an isomorphism, there exist unique elements $x_1,x_2 \in \Gamma_1$ such that $f(x_1) = y_1$, $f(x_2) = y_2$. Moreover, $f^{-1}(y_1+y_2) = f^{-1}(y_1) + f^{-1}(y_2)$, and $f^{-1}(y_1y_2) = f^{-1}(y_1)f^{-1}(y_2)$. Hence,

$$\begin{aligned}\widetilde{M}_{f(\Delta)}(y_1+y_2) &= \widetilde{M}_{\Delta}\left(f^{-1}(y_1+y_2)\right)\\ &= \widetilde{M}_{\Delta}\left(f^{-1}(y_1)+f^{-1}(y_2)\right)\\ &\geq \text{rmin}\{\widetilde{M}_{f(\Delta)},(y_1)\widetilde{M}_{f(\Delta)}(y_2)\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\widetilde{M}_{f(\Delta)}(y_1y_2) &= \widetilde{M}_{\Delta}\left(f^{-1}(y_1)f^{-1}(y_2)\right)\\ &\geq \text{rmax}\{\widetilde{M}_{f(\Delta)},(y_1)\widetilde{M}_{f(\Delta)}(y_2)\}.\end{aligned}$$

Also,

$$\begin{aligned}\widetilde{N}_{f(\Delta)}(y_1+y_2) &= \widetilde{N}_{\Delta}\left(f^{-1}(y_1+y_2)\right)\\ &\leq \text{rmax}\{\widetilde{N}_{f(\Delta)},(y_1)\widetilde{N}_{f(\Delta)}(y_2)\},\end{aligned}$$

and

$$\begin{aligned}\widetilde{N}_{f(\Delta)}(y_1y_2) &= \widetilde{N}_{\Delta}\left(f^{-1}(y_1)f^{-1}(y_2)\right)\\ &\leq \text{rmin}\{\widetilde{N}_{f(\Delta)},(y_1)\widetilde{N}_{f(\Delta)}(y_2)\}.\end{aligned}$$

Finally,

$$\begin{aligned}\widetilde{M}_{f(\Delta)}(y+y) &= \widetilde{M}_{\Delta}\left(f^{-1}(y+y)\right)\\ &= \widetilde{M}_{\Delta}\left(f^{-1}(y)+f^{-1}(y)\right)\\ &= \widetilde{M}_{\Delta}\left(f^{-1}(y)\right)\\ &= \widetilde{M}_{f(\Delta)}(y),\end{aligned}$$

and similarly, $\widetilde{N}_{f(\Delta)}(y+y) = \widetilde{N}_{f(\Delta)}(y)$. Therefore, $f(\Delta)$ satisfies all the defining conditions of a strong interval-valued intuitionistic fuzzy ideal of Γ_2. Hence, $f(\Delta)$ is a strong interval-valued intuitionistic fuzzy ideal of Γ_2.

Conclusion

In this paper, we introduce and explore the notion of a strong interval-valued intuitionistic fuzzy ideal within a semiring. This new definition enhances the current framework of interval-valued intuitionistic fuzzy ideals by enforcing stricter criteria on the interval-valued membership and non-membership functions. The literature has established several key properties, examples, lemmas, and theorems to elucidate the algebraic characteristics of these ideals. Notably, findings related to finite sums and products, arbitrary intersections, level subsets, and the maintenance of strong interval-valued intuitionistic fuzzy ideals through semiring homomorphisms and isomorphisms have been achieved. These insights offer a more profound comprehension of the structure of interval-valued intuitionistic fuzzy ideals in semirings and contribute to the broader theory of fuzzy algebras. The proposed concept could provide a valuable basis for further exploration of fuzzy algebraic structures, lattice theory, and applications dealing with uncertainty and decision-making. Future studies might concentrate on prime and maximal strong interval-valued intuitionistic fuzzy ideals, quotient semirings, and their relevance to other algebraic systems like rings, near-rings, and semirings..

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