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# Contraction and Fixed Point in Closed Subset of Metric Space.

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**Abstract**
The study of fixed point attributes and conditions on self-mappings and the complete set X that admits the existence of one or more fixed points make up fixed point theory. In the nonlinear analysis problem, In this paper we study about the  fixed point with some new contraction.The study of fixed point theorems is the generalization of fixed-point theorems existing in the literature.....
**Keywords:** Contraction Mapping, metric spaces, uniqueness, coincidence point fixed point, commuting mapping..

## Introduction

In the 20th century, Maurice Fréchet formally proposed the idea of a metric space, which laid the groundwork for contemporary topology and analysis. Since then, metric spaces have emerged as a key topic in mathematical research. When Stefan Banach developed the renowned Banach Contraction Principle in 1922, fixed point theory made significant strides.

Due to its numerous applications in mathematics, engineering, economics, computer science, and applied sciences, fixed point theory is one of the most significant and extensively researched fields of nonlinear analysis. The basic goal of the theory is to identify the circumstances in which a mapping defined on a particular space admits a point that is invariant under the mapping's action. Because they offer a natural framework for measuring distances and assessing convergence, metric spaces are essential among the many mathematical frameworks used to explore fixed points.

## METRIC SPACE

A metric space Y is a set together with a function s, called a metric that defines a distance between any two elements in Y. Formally, a metric space is a pair $(Y, s)$, where:

Y is a non-empty set.

$s : Y \times Y \to R^{+}$ is a function that assigns a non-negative real number to each pair of elements in Y.

The function s must satisfy the following properties for all $x, y, z \in Y$:

$$s(x, y) \geq 0$$

$$s(x, y) = 0 \Leftrightarrow x = y$$

$$s(x, y) = s(y, x)$$

$$s(x, y) \leq s(x, z) + s(z, y)$$

The introduction of cyclic contractions has enriched fixed point theory by allowing the analysis of mappings defined on the union of two or more subsets rather than on the entire space. This approach has proved particularly useful in studying problems where the domain naturally decomposes into multiple components. Furthermore, the notion of generalized contractions, such as the α–ψ contractions introduced by Samet et al [1]., has further broadened the scope of fixed-point results in metric spaces. The primary objective of this chapter is to present a systematic study of fixed point results for cyclic mappings in ordinary metric spaces.

We begin by recalling essential definitions and classical results related to metric spaces and cyclic contractions. Building upon these preliminaries, several generalized forms of cyclic contractions are introduced, and corresponding fixed-point theorems are established. These results not only unify but also extend to many well-known fixed-point theorems available in literature. The findings presented in this chapter contribute to the ongoing development of fixed-point theory by providing new generalized contraction conditions and demonstrating their effectiveness in ensuring the existence and uniqueness of fixed points. The results obtained here serve as a foundation for further generalizations and applications in more abstract metric-like spaces discussed in subsequent chapters.

Agrawal and Sushama [3], M. A. Al-Thafai and N. Shahzad [1] and Karapinar [2] presented numerous new theorems on cyclic contraction maps. Among the Banach contraction principle's generalizations, $\alpha - \varphi$ contraction is another generalization. Samet et al. [4] started it by proving fixed point theorems in entire metric spaces and introducing $\alpha - \varphi$ contraction type mappings. Only B. Samet [5] made additional improvements and modifications to this idea. After the paper of Samet et al. the $\alpha - \varphi$ contraction is used by many researchers in different metric spaces.

**SOME IMPORTANT DEFINITIONS**

Let M and N two non-empty subsets of a metric space $(Y, s)$.A mapping $g : M \cup N \to M \cup N$ is called **cyclic** if $g(M) \subset N$ and $g(N) \subset M$.

2) Let M and N be two non-empty subsets of a metric space$(Y, s)$. A mapping $h : M \cup N \to M \cup N$ is called cyclic contraction if there exist $t \in (0,1)$ such that $d(hx, hy) \le ts(x, y)$ for all $x \in M$ and $y \in N$

**3)** Let Mand N be non-empty closed subsets of a complete metric space $(Y, s)$. If a mapping $h : M \cup N \to M \cup N$ be cyclic contraction in Y then h has a unique fixed point in $M \cap N$.

4)Let M and N be non-empty subsets of a metric space $(Y, s)$. A mapping $g: M \cup N \to M \cup N$ is called Kannan type cyclic contraction if there exist

$$t \in (0, \tfrac{1}{2}) \text{ such that } s(gx, gy) \le t[s(gx, x) + s(gy, y) \text{ for all } x \in M \text{ and } y \in N$$

Let M and N are two non-empty closed subsets of a complete metric space $(Y, s)$.If a mapping $g : M \cup N \to M \cup N$ be Kannan type cyclic contraction in Y then g has a unique fixed point in $M \cap N$.

Let M and N are two non-empty subsets of a metric space$(Y, s)$. A mapping $g : M \cup N \to M \cup N$ is called Chetterjee type cyclic contraction if there exist

$t \in (0, \frac{1}{2})$ such that $s(gx, gy) \le t[s(fx, y) + s(gy, x)$ for all $x \in M$ and $y \in N$.

Let M and N are two non-empty closed subsets of a complete metric space $(Y, s)$.If a mapping $g : M \cup N \to M \cup N$ be Chetterjee type cyclic contraction in Y then g has a unique fixed point in $M \cap N$.

8)Let M and N are two non-empty subsets of a metric space$(Y, s)$. A mapping $g : M \cup N \to M \cup N$ is called Reich type cyclic contraction if there exists

$t \in (0, \frac{1}{3})$ such that $s(gx, gy) \le t[s(x, y) + s(gx, x) + s(gy, y)$ for all$x \in M$ and $y \in N$.

9) Let M and N are two non-empty closed subsets of a complete metric space $(Y, s)$.If a mapping $g : M \cup N \to M \cup N$ be Reich type cyclic contraction in Y then g has a unique fixed point in $M \cap N$.

On the basis above mentioned preliminaries, we conclude the following generalization of fixed point for cyclic mapping in complete metric spaces.

10) Let M and N are two non-empty subsets of a metric space (Y, s). A mapping $g : M \cup N \to M \cup N$ is called Generalized cyclic contraction if there exist

$i, j, k \in (0,1)$ satisfying $i + 2j + 2k < 1$ such that,

$$s(gx, gy) \le is(x, y) + j[s(gx, x) + s(gy, y)] + k[s(gx, y) + s(gy, x)]$$

For all $x \in M$ and $y \in N$.

**THEOREM:** Let M and N be non-empty closed subsets of a complete metric space (Y, s). If a mapping $g : M \cup N \to M \cup N$ be generalized cyclic contraction in Y then g has a unique fixed point in $M \cap N$.

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***Proof-*** Fix $x \in$ M. Now by the above definition there exist $i, j, k, \in (0,1)$ with $i + 2j + 2k < 1$ such that,

$$s(g^2x, gx) = s(g(gx), gx)$$
$$\leq is(gx, x) + j[s(g^2x, gx) + s(gx, x)] + k[s(g^2x, x) + s(gx, gx)]$$

$$\leq is(gx, x) + j[s(g^2x, gx) + s(gx, x)] + k[s(g^2x, gx) + s(gx, x)]$$

$$\leq (i + j + k)s(gx, x) + (i + k)s(g^2x, gx)\ [1 - (j + k)]s(g^2x, gx) \leq (i + j + k)s(gx, x)$$

So,
$s(g^2x, gx) \leq \frac{i+j+k}{1-(j+k)} s(gx, x)$. Then we get

$$(g^2\, x, gx) \leq \sigma s(gx, x)$$

$$\text{Where } \sigma = \frac{i + j + k}{1 - (j + k)} < 1$$
$$[\text{since } i + 2j + 2k < 1 \Rightarrow i + j + k < 1 - (j + k)\,]$$

By induction we can have $s(g^{n+1}x, g^n\, x) \leq \sigma\, ns(gx, x)$.Now for any $m, n \in N$ where $n > m$ we have

$$s(g^n\, x\,, g^m\, x)\ s(g^n\, x, g^{n-1}\, x) + s(g^{n-1}\, x, g^{n-2}\, x) \ldots \ldots \ldots + s(g^{m+1}\, x, g^m\, x)$$
$$\leq (\sigma\, n - 1 + \sigma\, n - 2 + \cdots + \sigma m + 1 + \sigma m)s(gx, x)$$
$$\leq \sigma\, m(\sigma\, n - m - 1 + \sigma\, n - m - 2 + \cdots + \sigma + 1)s(gx, x)$$
$$\leq \sigma\, m \frac{1}{1-\sigma} s(gx, x) \qquad [\text{Sum of infinite series as } n\,, m \to \infty\,]$$

Since $\sigma < 1$so $\sigma\, m \to 0$ as $n, m \to \infty$ thus, we have $s(g^n x, g^m x) \to 0$ as $n, m \to \infty$ therefore $\langle g^n x\rangle$ is Cauchy sequence. Since $(Y, s)$ is complete so $\langle g^n x\rangle$ converges to some point $p \in Y$ . Since $x \in M$ so $g^{2n}x \in M$ and $g^{2n-1}x \in N$ . But $g^{2n}x \to p$
and $g^{2n-1}x \to p$ . As the sets are closed so $p \in M \cap N$. It remains to show that
$gp = p$ we have,
$s(gp, g^{2n}x\,) = s(gp, g(g^{2n-1}x\,) \leq is(p, g^{2n-1}x\,) + j[s(gp, p) + x\,, g^{2n-1}x\,)] + k[s(gp, g^{2n-1}x\,) + s(g^{2n}x\,, p)]$
Taking lim. $n \to \infty$ we get $g^{2n}x = p$ and $g^{2n-1}x = p$ Then we have

$$s(gp, p) \leq is(p, p) + j[s(gp, p) + s(p, p)] + k[s(gp, p) + s(p, p)]$$
$$s(gp, p) \leq (j + k)s(gp, p)[\because g(p, p) = 0\,]$$
$$\Rightarrow [1 - (j + k)]s(gp, p) \leq 0$$

Since $j + k < 1$ so $s(gp, p) = 0 \Rightarrow gp = p$
Thus $p$ is a fixed point.To show the uniqueness of fixed point, let us suppose that there exist two fixed points $p$ and $q$ such that $gp = p$ and $gq = q$ Then we have

$$s(gp, gq) \leq is(p, q) + j[s(gp, p) + s(gq, q)] + k[s(gp, q) + s(gq, p)]$$
$$\leq i \leq is(p, q) + j[s(p, p) + s(q, q)] + k[s(p, q) + s(q, p)]$$
$$s(p, q) + 2ks(p, q)$$
$$[1 - (i + 2k)]s(p, q) \leq 0$$
$$\Rightarrow s(p, q) = 0[\because 1 - (i + 2k) > 0]$$
$$\Rightarrow p = q.$$

Therefore g has a unique fixed point in $M \cap N$ , this complete the proof.

**Conclusion on above theorems and some results**

The above theorem becomes the cyclic contraction theorem for $j = k = 0$.
The above theorem becomes the Kannan type cyclic contraction theorem for
$i = k = 0.1$.
The above theorem becomes the Chatterjee type cyclic contraction theorem when $i = j = 0$.
The above theorem becomes the Reich type cyclic contraction theorem when $i = j\ and\ k = 0$..

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