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The Upper Triangle Free Detour Domination Number and the Forcing Triangle Free Detour Domination Number of a Graph

[1]T.M.Selvarajan, [2]Sheeba.G
[1]Associate Professor, Department of Mathematics,
Noorul Islam Centre for Higher Education, Kumarakovil, Thuckalay-629180, India.
email:selvarajan1967@gmail.com
[2]Research Scholar, Department of Mathematics,
Noorul Islam Centre for Higher Education, Kumarakovil, Thuckalay-629180, India.
email:sheebaben97@gmail.com

Abstract

A minimum triangle free detour dominating set of G is a triangle free detour dominating set of S if no appropriate subset of S is a detour triangle dominating set of G. The upper triangle free detour domination number, indicated by $\gamma^{+}_{\mathrm{dn\Delta f}}(\mathrm{G})$, is defined as the highest cardinality of a minimal triangle free detour domination set of G. This concept's general qualities are investigated. The upper triangle free detour domination number of a family of graphs is calculated. It is demonstrated that for every pair of positive integers $3 \leq \mathrm{a} \leq \mathrm{b}$, there exists a connected graph G such that $\gamma_{\mathrm{dn\Delta f}}(\mathrm{G}) = \mathrm{a}$ and$\gamma^{+}_{\mathrm{\Delta fdn}}(\mathrm{G}) = \mathrm{b}$, where $\gamma_{\mathrm{\Delta fdn}}(\mathrm{G})$ is G's triangle free detour dominance number.This notion is investigated for several general features. A graph family's forced triangle free domination number can be computed. The maximum bipolar fuzzy spanning tree is used to introduce bipolar fuzzy detour g-interior and g-boundary nodes in a bipolar fuzzy tree. It is demonstrated that for every pair of positive integers a and b with $0 \leq \mathrm{a} \leq \mathrm{b}$, there exists a connected graph G such that $\mathrm{f}_{\gamma\mathrm{\Delta fdn}}(\mathrm{G}) = \mathrm{a}$ and, where $\gamma_{\mathrm{\Delta fdn}}(\mathrm{G}) = \mathrm{b}\gamma_{\mathrm{\Delta fdn}}(\mathrm{G})$ is G's triangular free domination number.

Keywords: distance, detour distance, trianglefree domination number, trianglefree detour domination number, forcing triangle free domination number.
AMS Subject Classification: 05C12, 05C69

1.Introduction

By a graph $G = (V, E)$, we mean a finite,undirected connected graph without loops or multiple edges. For basic graph theoretic terminology, we refer to [1]For the *neighborhood* of the vertex v in $G, N(v) = \{u \in V(G) : uv \in E(G)\}$. The *degree* of a vertex v of a graph is $deg(v) = |N(V)|$. $\Delta(G)$ and $\delta(G)$ are the maximum and minimum degrees of the graph respectively. A vertex v is said to be a *universal vertex* if $deg(v) = n - 1$. Let uv be an edge of G such that $deg(G) = 1$. Then G is called an *end vertex* and u is the *support vertex* of G. For $S \subseteq V(G)$, the *induced subgraph*$G[S]$ is the graph whose vertex set is S and whose edge set consists of all of the edges in E that have both endpoints in S. A vertex is called a *simplicial* vertex if the subgraph induced by its neighbors is complete.

The *distance* $d(u, v)$ between two vertices $u$and v in a connected graph G is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called an $u - v$*geodesic*.The *distanceD* (u,v) is the length of a longest u–v path in G. Au–v path of length $D(u,v)$ is called a u–v*detour*.The detour distance of a graph is studied in [4].A vertex x of a connected graph G is said to be a *detour simplicial vertex*[8] of G if x is not an internal vertex of any $u - v$ detour path for every $u, v \in V$. Each end vertex of G is a detour and v. si$I_D[u, v]$ consists of all vertices lying on some $u - v$detour of G including the vertices u mplicial vertex of G. Infact there are detour simplicial vertices which are not end vertices of G.The closed interval For $S \subseteq V$, $I_D[S] = \cup_{u,v \in S} I_D[u, v]$. A set S of vertices is called a detour set if $I_D[S] = V$. The *detour number* of $G$is the minimum order of its detour set of G and denoted by $dn(G)$. Any detour set of order $dn(G)$ is dn-set of G. The detour number of a graph is studied in [2,3,4, 6, 7, 8,10]. A path P is called a *triangle free path* if no three vertices of P induce a triangle. For vertices u and v in a connected graph G, *the triangle free detour distance* D△f(u, v) is the length of the longest u - v triangle free path in G. A u - v path of length D△f(u, v) is called a u-v *triangle free detour*. The concept of triangle free detour distance was introduced by Keerthi Asir et.al [6]. For a connected graph G, a set S ⊆ V is called a *triangle free detour set* of G if every vertex of G lies on a triangle free detour joining a pair of vertices of S. The *triangle free detour number*△f dn(G) of G is the minimum order of its triangle free detour sets and any triangle free detour set of order △f dn(G) is called a △f dn-set of G. These concepts were studied [9,10].

A set of vertices S in G is called a dominating set of Gif every vertex in $V \setminus S$ is adjscent to atleast one vertex of S.The *domination number* of G is the minimum cardinality of a dominating set of G and is denoted by $\gamma(G)$. A dominating set of size $\gamma(G)$ is said to be a γ*-set* of G. The *domination number* of a graph is studied in [8,9]. A set of vertices S in G is a triangle free detour dominating set or simply a $(\triangle fdn, D)$-set if S is both a triangle free detour set called and a dominating set. The minimum cardinality of a triangle free detour dominating set of G is its *triangle free detour domination number* of G or simply $(\triangle fdn, D)$ number of G and is denoted by $\gamma_{\triangle f}(G)$. Any $(\triangle fdn, D)$-set of size $\gamma_{\triangle fdn}(G)$ is said to be a $\gamma_{\triangle fdn}$-set of G.

The bipolar fuzzy graph and its numerous sorts of operations, as well as a specific type of bipolar fuzzy graph product, the concept of a strong arc in a fuzzy graph, and the distinct types of arc in a fuzzy graph Rosenfeld invented the notions of bridges, trees, cycles, cut node, and end node. The references explain several types of fuzzy graphs, as well as their function and use. The principles of triangle free detour domination have fascinating implications in radio technology channel assignment difficulties. These ideas can also be applied to the building of secure communication networks. The following theorem is implemented in the sequel.

Theorem 1.1[15]. Let$G$be a connected graph of order $n \geq 4$ such that $G \neq K_{1,n-1}$. Then G contains at least two detour simplicial vertices.
Theorem 1.2[15]. Each detour simplified vertex of G belongs to every detour triangle free dominating set of G.
Theorem 1.3[15]. Let G be a connected graph of order $n \geq 2$. Then $\gamma_{\Delta fdn}(G) = n$ if and only if $G = K_2$.
Theorem 1.4[15]. Let G be a connected graph of order $n \geq 3$. Then $\gamma_{\Delta fdn}(G) = n - 1$ if and only if $G = K_{1,n-1}$.

2.The Upper Detour Triangle Domination Number of a Graph

2.1. A **Definition** triangle dominating set S is called a minimal *detour triangle dominating set* of G if no proper subset of S is detour triangledominating setof G. The *upper detour triangle domination number*, denoted by $\gamma^{+}_{\Delta fdn}(G)$ is defined as the maximum cardinality of a minimal triangle free detour domination set of G.

Example 2.2. For the graph$G$ given in Figure 2.2, $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_1, v_3, v_5\}, S_3 = \{v_3, v_5, v_7\}, S_4 = \{v_3, v_7, v_6\}, S_5 = \{v_3, v_4, v_7\}$and $S_6 = \{v_1, v_3, v_4, v_6\}$, and are the only six minimal triangle free detour dominating sets of$G$ so that $\gamma^{+}_{\Delta fdn}(G) = 4$.

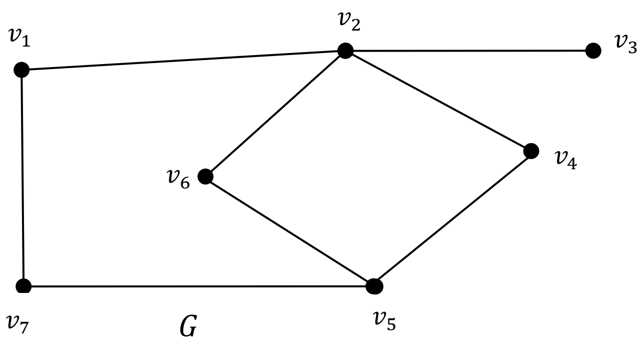


Figure 2.1

2.3. Note Every minimum triangle free detour dominating set of G is a minimal triangle free detour dominating set of G, but the converse is not true. For the graph G given in Figure 2.1, $S_6 = \{v_1, v_3, v_4, v_6\}$is a minimal triangle free detour dominating sets of G but it is not a minimum detour triangle dominating set of G.

Observation2.4.(i)Every detour simplicial vertex of Gbelongs to every triangle free detour dominating set of G.

(ii) Let G be a connected graph and v a cut vertex of G. Then every triangle free detour dominating set of G contain at least one vertex from each component of $G - v$.

(iii) Let G be a connected graph of order $n \geq 2$. Then $2 \leq \gamma_{\Delta fdn}(G) \leq \gamma^{+}_{\Delta fdn}(G) \leq n$.

2.5. (i)For the **Theorem** star $G = K_{1,n-1}, \gamma^{+}_{\Delta fdn}(G) = n$.

(ii)For the complete graph $G = K_n$, $\gamma^{+}_{\Delta fdn}(G) = 2$

(iii) For the complete bipartite graph $G = K_{m,n}(2 \leq m \leq n), \gamma^{+}_{\Delta fdn}(G) = 2$.

Proof: This follows from Observation 2.4(i) and (ii).

(ii) Let $V(K_n)= \{v_1, v_2, v_3, \ldots\}$. Then $S_{ij} = \{v_i, v_j\}$, $(1 \leq i,j \leq n)$ is a minimal triangle free detour dominating set of G and so $\gamma^{+}_{\Delta fdn}(G) \geq 2$. We prove that $\gamma^{+}_{\Delta fdn}(G) = 2$. On the contrary suppose that $\gamma^{+}_{\Delta fdn}(G) \geq 3$. Then there exists a triangle free detour dominating set S of G such that $|S| \geq 3$. Hence it follows that $S_{ij} \subset S$ for i,j $(1 \leq i,j \leq n)$, which is a contradiction. Therefore $\gamma^{+}_{\Delta fdn}(G) = 2$.

(iii)Let $U = \{v_1, v_2, \ldots, v_m\}$and $W = \{w_1, w_2, \ldots, w_n\}$ be two bipartite sets of G. Then $S_{ij} = \{u_i, w_j\}$, $(1 \leq i \leq m)$ $(1 \leq j \leq n)$ is a minimal triangle free detour dominating set of G and so $\gamma^{+}_{\Delta fdn}(G) \geq 2$. We prove that $\gamma^{+}_{\Delta fdn}(G) = 2$. On the contrary suppose that $\gamma^{+}_{\Delta fdn}(G) \geq 3$. Then there exists a triangle free detour dominating set S of G such that $|S| \geq 3$. Hence it follows that $S_{ij} \subset S$ for some i and $j(1 \leq i \leq m)$ $(1 \leq j \leq n)$, which is a contradiction. Therefore $\gamma^{+}_{\Delta fdn}(G) = 2$. ■

Theorem2.6. Let G be a connected graph of order $n \geq 2$ such that $G \neq K_2$. Then $\gamma^{+}_{\Delta fdn}(G) \leq n - 1$.

Proof. Since$G \neq K_2$, we have $n \geq 3$and G contains at least one non detour simplicial vertex. Let v be a non detour simplicial vertex of G. Then $S = V - \{v\}$ is a minimal triangle free detour dominating set of G so that $\gamma^{+}_{\Delta fdn}(G) \leq n - 1$.■

Theorem2.7. For a connected graph G of order $n \geq 2$, the following are equivalent:

(i) $\gamma^{+}_{\Delta fdn}(G) = n$

(ii) $\gamma_{\Delta fdn}(G) = n$

(iii) $G = K_2$.

Proof.$(i) \implies (ii)$.Let $\gamma^{+}_{\Delta fdn}(G) = n$. Then $S = V(G)$ is the unique minimal triangle free detour dominating set of G. Since no proper subset of S is a triangle free detour dominating set, it is clear that S is the unique minimum triangle free detour dominating set of G and so$\gamma_{\Delta fdn}(G) = n$. $(ii) \implies (iii)$.Let$\gamma_{\Delta fdn}(G) = n$. If $G \neq K_2$, then by Theorem 2.6,$\gamma_{\Delta fdn}(G) \leq n - 1$, which is a contradiction. Therefore $G = K_2$. (iii) $\implies (i)$.Let $G = K_2$. Then by Theorem 2.5(ii),$\gamma^{+}_{\Delta fdn}(G) = n$. ■

Theorem2.8. Let G be a connected graph of order $n \geq 3$such that $G \neq K_{1,n-1}$.Then $\gamma^{+}_{\Delta fdn}(G) \leq n - 2$.

Proof. Suppose that $\gamma^{+}_{\Delta fdn} \geq n - 1$. Then by Theorem 2.6,$\gamma^{+}_{\Delta fdn}(G) = n - 1$. Let v be a vertex of G and let $S = V - \{v\}$be the minimal triangle free detour dominating set of G. By Observation 2.4(i), v is not a detour simplicial vertexof G. Since $G \neq K_{1,n-1}$,there exists another non detour simplicial vertex of G, say u.Then $S_1 = S - \{u, v\}$ is a triangle free detour dominating set of. Since $S_1 \subsetneq S$, S_1 is not a minimal triangle free detour dominating set of G, which is a contradiction . Therefore$\gamma^{+}_{\Delta fdn}(G) \leq n - 2$. ■

Theorem 2.9 Let$G$be a connected graph of order $n \geq 3$ which is not K_3. If $\gamma^{+}_{\Delta fdn}(G) = n - 1$, then G contains a cut vertex.

Proof. If $n = 3$, then $G = P_3$. So we have done. So assume that $n \geq 4$. By Theorem 1.1, G contains at least two non-detour simplicial vertices, say u and v. Then $S = V - \{u, v\}$ is a triangle free detour dominating set of G so that $\gamma^{+}_{\Delta fdn}(G) \leq n - 2$,which is a contradiction. Therefore$G$ contains a cut vertex. ■

Theorem 2.10. Let$G$be a connected graph of order $n \geq 4$ with $D \geq 3$. Then $\gamma^{+}_{\Delta fdn}(G) \leq n - 2$.

Proof. Suppose that $\gamma^{+}_{\Delta fdn}(G) \geq n - 1$. Then by Theorem 2.7, $\gamma^{+}_{\Delta fdn}(G) = n - 1$. Let$S = V - \{v\}$ be a minimum triangle free detour dominating set of G. Then by Observation 2.4(i) and Theorem 2.9,v is neither a cut vertex of G not a detour simplicial vertex set of G. Since $n \geq 4$, by Theorem 1.1,G contains another non-detour simplicial vertex, say u. Then $S_1 = S - \{u, v\}$ is a triangle free detour dominating set of G, which is a contradiction. Therefore $\gamma^{+}_{\Delta fdn}(G) \leq n - 2$.■

Theorem 2.11 . For a connected graph G of order $n \geq 3$, the following are equivalent:

(i) $\gamma^{+}_{\Delta fdn}(G) = n - 1$

(ii) $\gamma_{\Delta fdn}(G) = n - 1$

(iii) $G = K_{1,n-1}$ or $G = K_3$

Proof. $(i) \Longrightarrow (ii)$. Let $\gamma^{+}_{\Delta fdn}(G) = n-1$. Then it follows from Theorem 2.7 that $n \geq 3$. Hence by Theorem 2.9, G contains a cut vertex, say v. Since $\gamma^{+}_{\Delta fdn}(G) = n - 1$, hence it follows from Theorem 2.6 that $S = V - \{v\}$ is the unique triangle free detour dominating set of G. We claim that $\gamma_{\Delta fdn}(G) = n-1$. Suppose that $\gamma_{\Delta fdn}(G) < n-1$. Then there exists a minimum triangle free detour dominating set M_1 such that $|M_1| < n-1$. It is clear that $v \notin M_1$. Then it follows that $M_1 \underset{\neq}{\subset} M$, which is a contradiction. Therefore $\gamma_{\Delta fdn}(G) = n - 1$. $(ii) \Longrightarrow (iii)$. Let $\gamma_{\Delta fdn}(G) = n-1$. Then by Theorem2.10, $D \leq 2$. By Theorem 2.7, $D = 2$. Then G is either K_3 or $K_{1,n-1}$. So we have done. ■

Theorem 2.12. For the non-trivial path $G = P_n (n \geq 4)$, $\gamma^{+}_{\Delta fdn}(G) = n-2$.

Proof. Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$. Then $S = V(P_n) - \{x, y\}$ is a minimal triangle free detour dominating set of G and so $\gamma^{+}_{\Delta fdn}(G) \geq n-2$, where $xy \in E(G)$ such that x and y are not end vertices of G. We prove that $\gamma^{+}_{\Delta fdn}(G) = n-2$. On the contrary suppose that $\gamma^{+}_{\Delta fdn}(G) \geq n-1$. Then by Theorem $\gamma^{+}_{\Delta fdn}(G) = n-1$, Let T be a minimal triangle free detour dominating set of G with $|T| = n-1$. Then $S \subset T$, which is a contradiction. Therefore $\gamma^{+}_{\Delta fdn}(G) = 2$.

Theorem 2.13. For each pair a and b of integers $3 \leq a \leq b$, there exists a connected graph G with $\gamma_{\Delta fdn}(G) = a$ and $\gamma^{+}_{\Delta fdn}(G) = b$.

Proof. If $a = b$, $G = K_{1,a-1} (a \geq 4)$ has the desired properties. Assume that $3 \leq a \leq b$. Let $F = K_2 \cup [(b-1) + 2K_1] + \overline{K}_2$, where let $X = V(K_2) = \{x_1, x_2\}$, $Y = V[(b-a) + 2K_1] = \{x, v_1, v_2, \dots, v_{b-a+2}\}$ and $Y = V(\overline{K}_2) = \{y_1, y_2\}$. Let G be the graph obtained from F by adding $a-2$ new vertices $w_1, w_2, \dots, w_{a-2}$ and joining each w_i to x. The graph G is shown in Figure 2.1. Let $W = \{w_1, w_2, \dots, w_{a-2}\}$ be the set of end vertices of G.

First, we show that $\gamma_{\Delta fdn}(G) = a$. By Observation 2.4 (i), $\gamma_{\Delta fdn}(G) \geq a - 2$. It is easily observed that W is not a triangle free detour dominating set of G. Also it is easily seen that $W \cup \{u\}$ is not triangle free detour dominating set of G, where $u \notin S$ and so $\gamma_{\Delta fdn}(G) \geq a$. On the other hand, let $S = \{w_1, w_2, \dots, w_{a-2}, x_1, y_1\}$. Then S is a triangle free detour dominating set of G. Therefore, $\gamma_{\Delta fdn}(G) = a$.

Next we show that $\gamma^{+}_{\Delta fdn}(G) = b$. Let $= \{u_1, w_1, w_2, \dots, w_{a-2}, v_1, v_2, \dots, v_{b-a+1}\}$. It is clear that T is a triangle free detour dominating set of G. We claim that T is a minimal triangle free detour dominating set of G. Assume, to the contrary, that T is not a minimal triangle free detour dominating set. Then there is a proper subset M of T such that M is an detourtriangle dominating set of G. Let $u \in T$ and $u \notin M$. By Observation 2.4(i), $u \neq w_i$, for all $i = 1, 2, \dots, a-2$. If $u = v_i$ for some $i (i \leq i \leq b-a+1)$, then, $u \notin I_D[M]$, which is a contradiction. If $u = u_1$, then M is not a triangle free detour dominating set of G, which is a contradiction. Thus T is a minimal detourtriangle dominating set of G and so $\gamma^{+}_{\Delta fdn}(G) \geq a - 2 + b - a + 1 + 1 = b$. It is easily seen that $T_1 = \{u_2, w_1, w_2, \dots, w_{a-2}, v_1, v_2, \dots,$
$v_{b-a+1}\}$, $T_2 = \{u_1, w_1, w_2, \dots, w_{a-2}, v_1, v_2, \dots, v_{b-a+1}\}$, $T_3 = \{z_1, w_1, w_2, \dots w_{a-2}, v_1, v_2,$
$\dots, v_{b-a+1}\}$ and $T_4 = \{z_2, w_1, w_2, \dots, w_{a-2}, v_1, v_2, \dots, v_{b-a+1}\}$ are minimal triangle free detour dominating sets of G. We prove that $\gamma^{+}_{\Delta fdn}(G) = b$. On the contrary suppose that $\gamma^{+}_{\Delta fdn}(G) \geq b+1$. Then there exists a triangle free detour dominating set D of G such that $|D| \geq b+1$. Since $G[V-D]$ is connected, $x \notin D$. Then it follows that either S or T or T_1 or T_2 or T_3 or T_4 is a subset of D, which is a contradiction to D a minimal triangle free detour dominating sets of G. Therefore, $\gamma^{+}_{\Delta fdn}(G) = b$. ■

2.

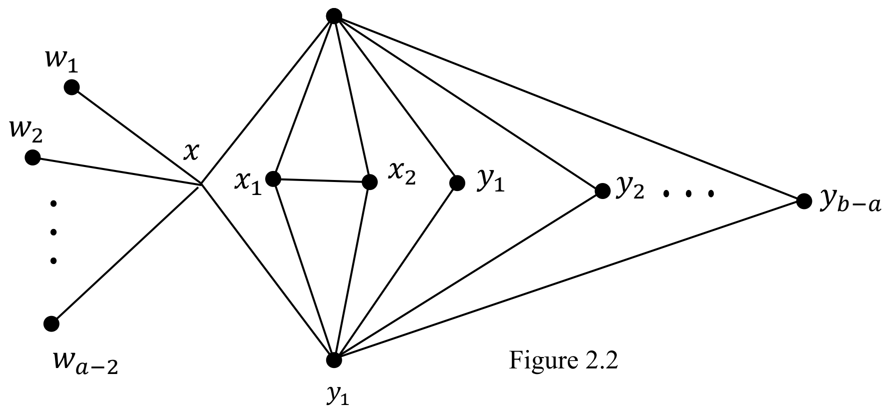


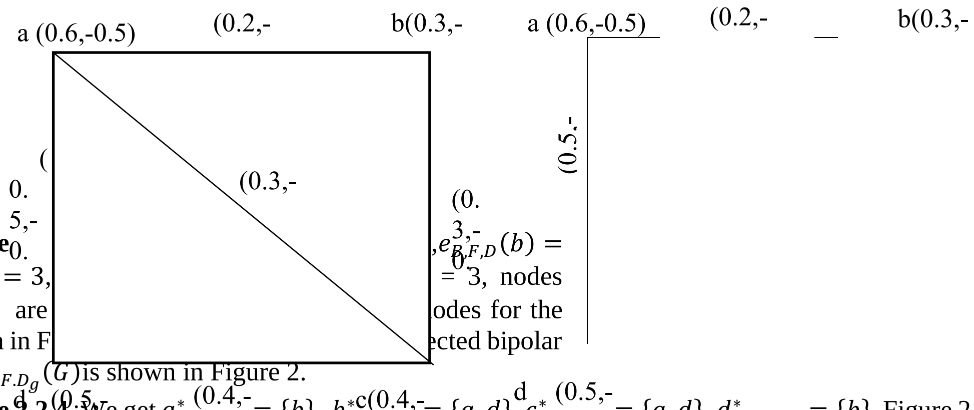
Figure 2.2

2.2.BIPOLAR FUZZY DETOUR G- DISTANCE

Definition 2.2.1: In a connected bipolar fuzzy graph$G = (V, A, B)$, the length of an $x\ y$ strong path P between x and y is said to be a bipolar fuzzy detour g-distance if there is no other strong path longer than P between x and y, and we denote this by$BFD_g(x, y)$. A xy bipolar fuzzy g-detour is any xy strong path whose length is $BFD_g(x, y)$.

Definition 2.2.1. A bipolar fuzzy detour g-periphery of G is a bipolar fuzzy subgraph of the bipolar fuzzy graph$G = (V, A, B)$, whose nodes are only the bipolar fuzzy detour g-peripheral nodes and is denoted by$per_{B.F.D_g}(G)$.

Definition 2.2.2G is said to be a bipolar fuzzy detour geocentric bipolar fuzzy graph if each node of a connected bipolar fuzzy graph G = (V, A, B) is a bipolar fuzzy detour g-eccentric node. A bipolar fuzzy detour g-eccentric bipolar fuzzy subgraph of G is denoted by $Ecc_{B.F.D_g}(G)$, and it is formed by the set of all bipolar fuzzy g-eccentric nodes of$G$.



Example ... $,e_{B,F,D}(b) =$ 3,$e_{B,F,D}(c) = 3,$ ... = 3, nodes $a, b, and\ d$ are ... odes for the bipolar fuzzy graph in F... cted bipolar fuzzy graph $per_{B.F.D_g}(G)$is shown in Figure 2.

Example 2.2.4. We get $a^*_{B,F,D_g} = \{b\}$, $b^*_{B,F,D_g} = \{a, d\}$, $c^*_{B,F,D_g} = \{a, d\}$, $d^*_{B,F,D_g} = \{b\}$. Figure 2 depicts its $Ecc_{B,F,D_g(G)}$.

3. TheForcing DetourTriangle Domination Number of a Graph

Definition3.1. For a minimum detour triangle dominating set S of G, a set $T \subseteq S$ is called a *forcing subset*of S if $S$is the unique minimum $\gamma_{\Delta fdn}$-set containing T. A forcing subset for $S$of minimum cardinality is a *minimum* *forcing subset of* S. The *forcing detour triangle domination number* of S,denoted by $f_{\gamma\Delta fdn}(S)$ is the cardinality of a minimum forcing subset of S. The *forcing triangle free detour domination number* of $G, f_{\gamma\Delta fdn}(G) = \min\{f_{\gamma\Delta fdn}(S)\}$, where the minimum is taken over all minimum$\gamma_{\Delta fdn}$-sets Sin G.

Example 3.2. For the graph G given in Figure 3.1,$S_1 = \{v_1, v_2, v_4\}$ and $S_2 = \{v_1, v_4, v_7\}, S_3 = \{v_2, v_6, v_4\}, S_4 = \{v_2, v_4, v_7\}, S_5 = \{v_4, v_5, v_7\}, S_6 = \{v_4, v_6, v_7\}, S_7 = \{v_3, v_4, v_6\}$ are the only seven$\gamma_{\Delta fdn}$-sets of G such that $f_{\gamma\Delta fdn}(S_1) = 3, f_{\gamma\Delta fdn}(S_2) = 2, f_{\gamma\Delta fdn}(S_3) = 2$, $f_{\gamma\Delta fdn}(S_4) = 2, f_{\gamma\Delta fdn}(S_5) = 2, f_{\gamma\Delta fdn}(S_6) = 2, f_{\gamma\Delta fdn}(S_7) = 2$ so that $f_{\gamma\Delta fdn}(G) = 2$.

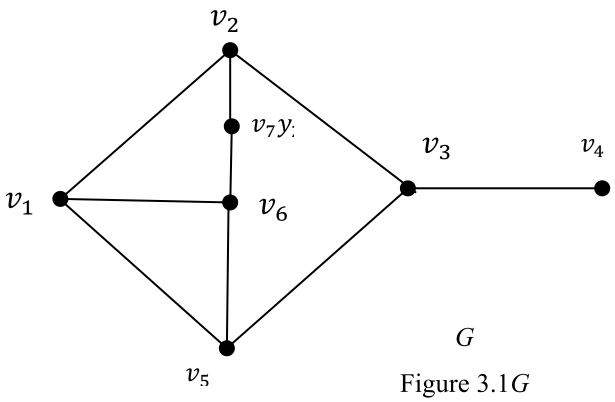


Figure 3.1G

Theorem3.3. For every connected graph, $0 \leq f_{\gamma\Delta fdn}(G) \leq \gamma_{\Delta fdn}(G)$.

The followingtheoremcharacterizes graphs $G$for which the bounds in Theorem2.3 are attained and also graphs for which$f_{\gamma\Delta fdn}(G) = 1$.The proof of the theoremisstraightforwardsowe omit the proof.

Observation 3.4. Let G be a connected graph. Then

(i)$f_{\gamma\Delta fdn}(G) = 0$If and only if G has a unique minimumtriangle free detour dominating set.

(ii) $f_{\gamma\Delta fdn}(G) = 1$ if and only if G has at least two minimum triangle free detour dominating sets, one of which is a unique minimum triangle free detour dominating setcontaining one of its elements, and

(iii) $f_{\gamma\Delta fdn}(G) = \gamma_{\Delta fdn}(G)$ if and only if no minimum triangle free detour dominating set of $G$is the unique minimum triangle free detour dominating set containingany of itspropersubsets.

Definition3.5. A vertex v of a graph Gissaid to be a *detour triangle free dominatingvertex*if $v$belongs to every minimum detour triangle dominating set of G.

Remark 3.6. For the graph$G$ given in Figure 3.1, v_4 is the set of all detour triangle free dominatingvertex of G.

Theorem3.7. Let G be a connected graph and W be the set of all detour trianglefreedominating vertices of G. Then$f_{\gamma\Delta fdn}(G) \leq \gamma_{\Delta fdn}(G) - |W|$.

Proof. Let S be any minimum triangle free detourdominating set of G. Then $\gamma_{\Delta fdn}(G) = |S|$, $W \subseteq S$ and S is the unique minimum detour triangle dominating set containing $S - W$. Thus $f_{\gamma\Delta fdn}(G) \leq |S - W| = |S| - |W| = \gamma_{\Delta fdn}(G) - |W|$. ∎

Theorem 3.8. For the star $G = K_{1,n-1}(n \geq 3), f_{\gamma\Delta fdn}(G) = 0$.

Proof. This follows from Theorems 1.1 and Observation 3.4 (i).

Theorem 3.9. For the non-trivial path $G = P_n(n \geq 3)$,

$$f_{\gamma\Delta fdn}(G) = \begin{cases} 0, & for\ n = 3, \\ 2, & for\ n = 4, \\ 1, & for\ n = 5,6, \\ \frac{n-3}{2}, & for\ odd\ n \geq 7, \\ \frac{n-2}{2}, & for\ even\ n \geq 8 \end{cases}$$

Proof. Let $P_n : v_1, v_2, \dots, v_{n-1}, v_n$. Let $n = 3$. Then $S_1 = \{v_1, v_3\}$, is the unique $\gamma_{\Delta fdn}$-set of G so that $f_{\gamma\Delta fdn}(G) = 0$. Let $n = 4$. Then $S = \{v_1, v_4\}$ is the unique $\gamma_{\Delta fdn}$-set of G so that $f_{\gamma\Delta fdn}(G) = 0$. Let $n = 5$. Then $S_1 = \{v_1, v_4, v_5\}$ and $S_2 = \{v_1, v_2, v_5\}$ are the only two $\gamma_{\Delta fdn}$-sets of G such that $f_{\gamma\Delta fdn}(S_1) = f_{\gamma\Delta fdn}(S_2) = 1$. Let $n = 6$. Then $S_1 = \{v_1, v_4, v_5, v_6\}, S_2 = \{v_1, v_2, v_5, v_6\}$ and $S_3 = \{v_1, v_2, v_3, v_6\}$ are the three $\gamma_{\Delta fdn}$-sets of G such that $f_{\gamma\Delta fdn}(S_1) = f_{\gamma\Delta fdn}(S_3) = 1$ and $f_{\gamma\Delta fdn}(S_2) = 2$ so that $f_{\gamma\Delta fdn}(G) = 1$.

For odd $n \geq 7$, there are $n-3$ $\gamma_{\Delta fdn}$-sets viz., $S_1 = \{v_1, v_4, v_5, \dots, v_n\}, S_2 = \{v_1, v_2, v_5, \dots, v_n\}$, $S_3 = \{v_1, v_2, v_3, v_6, \dots, v_n\}, \dots, S_{n-5} = \{v_1, v_2, v_3, \dots, v_{n-5}, v_{n-2}, v_{n-1}, v_n\}, S_{n-4} = \{v_1, v_2, v_3, \dots, v_{n-4}, v_{n-1}, v_n\}, S_{n-3} = \{v_1, v_2, v_3, \dots, v_{n-3}, v_n\}$. We observe that $T_1 = \{v_4, v_6, \dots, v_{n-1}\}$ is a minimum forcing subset of S_1 and so $f_{\gamma\Delta fdn}(S_1) = \frac{n-3}{2}$, $T_{n-5} = \{v_2, v_4, \dots, v_{\frac{n-1}{2}}, v_6, v_9, \dots, v_{n-2}\}$ is a minimum forcing subset of S_{n-5} and so $f_{\gamma\Delta fdn}(S_{n-5}) = \frac{n-3}{2}$, $T_{n-4} = \{v_3, v_6, \dots, v_{\frac{n+1}{2}}, v_7, v_{10}, \dots, v_{n-1}\}$ is a minimum forcing subset of S_{n-4} and so $f_{\gamma\Delta fdn}(S_{n-4}) = \frac{n-3}{2}$ and $T_{n-3} = \{v_2, v_4, v_6, \dots, v_{n-3}\}$ is a minimum forcing subset of S_{n-3} and so $f_{\gamma\Delta fdn}(S_{n-3}) = \frac{n-3}{2}$. For $i = 2, 4, 6, \dots, n-7$, $S_i = \{v_1, v_2, \dots, v_i, v_{i+3}, v_{i+4}, \dots, v_n\}$. Then $T_i = \{v_2, v_4, \dots, v_i, v_{i+3}, v_{i+5}, \dots, v_{n-2}\}$ is a minimum forcing subset of S_i and so $f_{\gamma\Delta fdn}(S_i) = \frac{n-3}{2}$. For $j = 3, 5, 7, \dots, n-6, S_j = \{v_1, v_4, \dots, v_i, v_{i+3}, v_{i+4}, \dots, v_n\}$. Then $T_j = \{v_3, v_5, \dots, v_{j+5}, \dots, v_{n-1}\}$ is a minimum forcing subset of S_j and so $f_{\gamma\Delta fdn}(S_j) = \frac{n-3}{2}$. Therefore $f_{\gamma\Delta fdn}(G) = \frac{n-3}{2}$ for odd $n \geq 7$.

For even $n \geq 8$, there are $n-3$ $\gamma_{\Delta fdn}$--sets viz., $S_1 = \{v_1, v_4, v_5, \dots, v_n\}, S_2 = \{v_1, v_2, v_5, \dots, v_n\}$, $S_3 = \{v_1, v_2, v_3, v_6, \dots, v_n\}, \dots, S_{n-5} = \{v_1, v_2, v_3, \dots, v_{n-5}, v_{n-2}, v_{n-1}, v_n\}, S_{n-4} = \{v_1, v_2, v_3, \dots, v_{n-4}, v_{n-1}, v_n\}, S_{n-3} = \{v_1, v_2, v_3, \dots, v_{n-3}, v_n\}$. Then $T_1 = \{v_4, v_6, \dots, v_n\}$ is a minimum forcing subset of S_1 and so $f_{\gamma\Delta fdn}(S_1) = \frac{n-2}{2}, T_{n-5} = \left\{v_3, v_5, \dots, v_{\frac{n}{2}+1}, v_{\frac{n-2}{2}}, \dots, v_{n-1}\right\}$ is a minimum forcing subset of S_{n-5} and so $f_{\gamma\Delta fdn}(S_{n-5}) = \frac{n-2}{2}, T_{n-4} = \left\{v_2, v_4, \dots, v_{\frac{n}{2}}, v_{\frac{n}{2}+2}, \dots, v_{n-1}\right\}$ is a minimum forcing subset of S_{n-4} and so $f_{\gamma\Delta fdn}(S_{n-4}) = \frac{n-2}{2}$ and $T_{n-3} = \{v_2, v_4, \dots, v_{\frac{n}{2}}, v_{\frac{n}{2}+2}, \dots, v_{n-3}\}$ is a minimum forcing subset of S_{n-3} and so $f_{\gamma\Delta fdn}(S_{n-3}) = \frac{n-2}{2}$. For $i = 2, 4, 6, \dots, n-6, S_i = \{v_1, v_2, \dots, v_i, v_{i+3}, v_{i+4}, \dots, v_n\}$. Then $T_i = \{v_2, v_4, \dots, v_i, v_{i+3}, v_{i+5}, \dots, v_{n-1}\}$ is a minimum forcing subset of S_i and so $f_{\gamma\Delta fdn}(S_i) = \frac{n-2}{2}$. For $j = 3, 5, 7, \dots, n-7, Sj = \{v_1, v_2, \dots v_i, v_{i+3}, v_{i+4}, \dots, v_n\}$. Then $T_j = \{v_2, v_4, \dots, v_{j-2}, v_j, v_{j+1}, v_{j+3}, \dots, v_{n-3}\}$

is a minimum forcing subset of S_j and so $f_{\gamma\Delta fdn}(S_j) = \frac{n-2}{2}$. Therefore $f_{\gamma\Delta fdn}(G) = \frac{n-2}{2}$ for even $n \geq 8$. ■

Theorem 3.10. For the complete bipartite graph $= K_{r,s}$ $(2 \leq r \leq s), f_{\gamma\Delta fdn}(G) = 2$.

Proof. Let $X = \{x_1, x_2, \dots, x_r\}$ and $Y = \{y_1, y_2, \dots, y_s\}$ be the bipartite sets of G. Let $S_{ij} = \{x_i, y_j\}$ $(1 \le i \le r)$ and $(1 \le j \le s)$. Then S_{ij} is a $\gamma_{\Delta fdn}$-set of G so that $\gamma_{\Delta fdn}(G) = 2$. Since any singleton subset of S_{ij} is a subset of more than one $\gamma_{\Delta fdn}$-set of G for some i and j. Therefore $f_{\gamma\Delta fdn}(G) \ge 2$. By Theorem 3.3, $f_{\gamma\Delta fdn}(G)$ = 2.■

Theorem 3.11. For the cycle $G = C_n$ $(n \ge 3)$, $f_{\gamma\Delta fdn}(G) = \gamma_{\Delta fdn}(G) = n - 2$.
Proof. Let $C_n : v_1, v_2, \dots, v_{n-1}, v_n$ be the cycle of order n. Then $S_1 = \{v_1, v_2, v_3, \dots, v_{n-2}\}, S_2 = \{v_2, v_3, v_4, \dots v_{n-2}, v_{n-1}\}, S_3 = \{v_3, v_4, \dots, v_{n-1}, v_n\}, \dots, S_n = \{v_n, v_1, v_2, v_3, \dots, v_{n-3}\}$ are the $\gamma_{\Delta fdn}$- sets of G with cardinality $n-2$. We notice that no $\gamma_{\Delta fdn}$-set of G is the unique $\gamma_{\Delta fdn}$-set containing any of its proper subsets. Therefore $f_{\gamma\Delta fdn}(G) = \gamma_{\Delta fdn}(G) = n - 2$. ■

Theorem 3.12. For every pair of positive integers a and b with $0 \le a \le b$ and $b \ge 2$, there exists a connected graph G such that $f_{\gamma\Delta fdn}(G) = a$ and $\gamma_{\Delta fdn}(G) = b$.

Proof. For $a = 0, b \ge 2$, let $G = K_{1,b}$, Then by Theorems 1.2 and 3.8, $f_{\gamma\Delta fdn}(G) = 0$ and $\gamma_{\Delta fdn}(G) = b$. So let $0 < a \le b$
Case1. $a = b$. Consider the graph $G = C_{a+2}$ $(a \ge 2)$. Then by Theorem 3.11, $f_{\gamma\Delta fdn}(G) = \gamma_{\Delta fdn}(G) = a$.
Case 2. $0 < a < b$. Let $P_i: u_i, v_i$ $(1 \le i \le a)$ be a copy of path on two vertices. Let G_a be the graph obtained from $P_i (1 \le i \le a)$ by adding new vertex x and introducing the edges xu_i and $xv_i (1 \le i \le a)$. Let G be the graph obtained from G_a by adding new vertices $z_1, z_2, \dots, z_{b-a}$ and introducing the edge $xz_i (1 \le i \le b-a)$. First we show that $\gamma_{\Delta fdn}(G)= b$. Let $Z = \{z_1, z_2, \dots, z_{b-a}\}$. By Theorem 1.1, Z is a subset of every minimum detour trianglefree dominating set of G. Let $H_i = \{u_i, v_i\} (1 \le i \le a)$. Then it is easily observed that every triangle free detourdominating set of G contains at least one vertex from each H_i $(1 \le i \le a)$ and so $\gamma_{\Delta fdn}(G) \ge b - a + a = b$. Let $S = Z \cup \{u_1, u_2, \dots, u_a\}$. Then S is a minimum detour triangle dominating set of G so that $\gamma_{\Delta fdn}(G) = b$.

Next we show that $f_{\gamma\Delta fdn}(G) = a$. By Theorem 2.7, $f_{\gamma\Delta fdn}(G) \le \gamma_{\Delta fdn}(G) - |Z| = b - (b - a) = a$. Now, since $\gamma_{\Delta fdn}(G) = b$ and Z is a subset of every minimum triangle free detour dominating free set of G, it is easily seen that every $\gamma_{\Delta fdn}$-set of G is of the form $S' = Z \cup \{c_1, c_2, \dots, c_a\}$, where $c_i \in H_i$ $(1 \le i \le a)$. Let T be any proper subset of S' with $|T| < a$. Then it is clear that there exists some i such that $T \cap H_i = \phi$, which shows that $f_{\gamma\Delta fdn}(G) = a$.

CONCLUSION

The upper triangle free detour domination number, represented by $\gamma^{+}_{dn\Delta f}(G)$, is the greatest cardinality of a minimal triangle free detour domination set of G. There is a connected graph G such that $\gamma_{dn\Delta f}(G) = a$ and $\gamma^{+}_{\Delta fdn}(G) = b$, where _fdn (G) for any pair of positive integers a and b with $3 \le a \le b$. G was also established as the forced triangle free detour dominating number. This notion is being looked into for certain generic qualities. The forcing triangle method is used to compute the free domination number of a family of graphs. We used detour g-distance, detour g-boundary nodes, and detour g-interior nodes, as well as their attributes, in bipolar fuzzy graphs. The forcing triangle method is used to compute the free domination number of a family of graphs. In bipolar fuzzy graphs, we used detour g-distance, detour g-boundary nodes, and detour g-interior nodes, as well as their attributes. We established theorems on the detour g-interior node, detour g-boundary node, and cut node in a bipolar fuzzy graph using the maximum bipolar fuzzy spanning tree. It is demonstrated that for every pair of positive integers a *and* b *with* $0\, a\, b$, a connected graph G exists with $f_{\gamma\Delta fdn}(G) = a$ and $\gamma_{\Delta fdn}(G) = b$, where $\gamma_{\Delta fdn}(G)$ is the triangle free domination number.

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